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PROCEEDINGS

OF THE

LONDON MATHEMATICAL SOCIETY.

VOL. VII.

FROM NOVEMBER, 1875, TO NOVEMBER, 1876.

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PROCEEDINGS

OF THE

LONDON MATHEMATICAL SOCIETY.

VOL. VII.

TWELFTH SESSION, 1875–6.

November 11th, 1875.

ANNUAL GENERAL MEETING, held at 22, Albemarle Street.

Prof. CAYLEY, F.R.S., in the Chair.

The Treasurer (Samuel Roberts, Esq., M.A.) read his Report, and Mr. Perigal, F.R.A.S., kindly consented to act as Auditor.

The Secretaries' Report was then read and adopted.

From the Report of the Secretaries it appeared that the number of Members had increased from 118 to 133 during the past Session, and that the present number of Life Members is 33.

The Obituary of the Society, during the Session, comprised the name of Mr. J. H. Röhrs, M.A., formerly Fellow of Jesus College, Cambridge.

The Communications made to the Society during the Session had been as follows:—

"Correlation in Space:" by Dr. Hirst, F.R.S.

"Tidal Retardation:" by Mr. J. H. Röhrs.

"On the Porism of the In- and Circum-scribed Triangle:" by Prof. Wolstenholme.

"The Potentials of Polygons and Polyhedra;" "The Potentials of an Ellipse and Circle;" "On the Attraction of an Ellipsoidal Shell;" "On a Point in the Theory of Attraction:" by Prof. Cayley, F.R.S.

"On Three and Seven-Bar Motion:" (two letters from M. Mannheim to Mr. J. J. Sylvester, F.R.S.)

"On the Solution of Linear Differential Equations in Series:" by Mr. J. Hammond.

"The Construction and Use of a Mechanical Calculator:" by Major J. R. Campbell.

"On the Representation of any Unicursal Curve and its Nodes in terms of the Parametric Coefficients, and on Roberts' cases of Unicursal Three-bar Motion:" by J. J. Sylvester, F.R.S.

"On the Question of the Mechanical Description of a Quartic Curve:" by Prof. Cayley.

"The Mode of Construction and Properties of a new sort of Lady's Fan;" and "On the Expression of the Curves generated by any given system whatever of Link-work under the form of an Irreducible Determinant:" by J. J. Sylvester, F.R.S.

"On Peaucellier's Problem:" by Rev. W. H. Laverty.

"On Laplace's Three Particles:" by Mr. E. J. Routh, F.R.S.

"On some Relations between certain Elliptic and Hyperbolic Functions:" by Mr. J. Griffiths.

"On a Simplified Method of obtaining the Order of Algebraical Conditions:" by Mr. S. Roberts.

"An Orthogonal Web:" by J. J. Sylvester, F.R.S.

"On some Proposed Forms of Slide-Rule;" "The Mechanical Description of Equipotential Lines;" and "A Mechanical Method of making a Force which varies inversely as the Square of the Distance from a Fixed Point:" by Mr. G. H. Darwin.

"The Integration of the Equations for the Motions of a System acted on by Forces expressed by Linear Functions of the Displacements and Velocities;" and "On the Vibrations of a Stretched String of Gyrostats (Dynamical Theory of Faraday's Magnetic Rotation of the Plane of Polarisation):" by Sir W. Thomson, F.R.S.

"On some Integrals connected with the Theory of Attraction:" by Prof. Cayley, F.R.S.

"A Hydrostatical Toy:" by Prof. Wolstenholme.

"On the application of Hamilton's Characteristic Function to Optical Instruments symmetrical about an Axis, and the Value of the Function for a Spherical Surface:" by Prof. J. Clerk-Maxwell, F.R.S.

"On some Constructions for Transforming Curves and Surfaces:" by Rev. C. Taylor.

"Notes on Laplace's Coefficients:" by Mr. J. W. L. Glaisher, F.R.S.

"On a Link-work for Describing Sphero-Conics and Sphero-Quartics:" by Mr. Harry Hart.

"On some Figures of Curves in Three-bar Motion:" by Prof. Cayley.

"On James Watt's Parallel Motion:" by J. J. Sylvester, F.R.S.

"On the Correspondence of Points Collinear with a Fixed Origin:" by Mr. T. Cotterill.

The Proceedings had been sent to the same persons and Societies, and the same Mathematical Journals had been subscribed for as in the preceding Session.

The following presents had been made to the Society:—

"Jahrbuch über die Fortschritte der Mathematik," vierter Band, Heft 2, 3. Jahrgang 1872; fünfter Band, Heft 1, 2, 3, Jahrgang 1873. Berlin, 1874.

"Physical Society of London Proceedings," Part 1, March 21 to June 20, 1874; Part 2, Nov. 7, 1874, to Jan. 18, 1875.

"Nautical Almanac for 1878."

"Memoir on the Transformation of Elliptic Functions;" by Prof. Cayley.

"Two Mechanical Calculators:" from Major Campbell.

"On Tautochronous and Brachystochronous Curves for Parallel and Concurrent Forces;" by the Rev. Prof. Townsend, F.R.S.

"On the Free Equilibrium of a Uniform Cord compared with the Free Motion of a Material Particle under the Action of a Central Force;" by the same.

"Remarks on the Great Logarithmic and Trigonometrical Tables," computed in the Bureau du Cadastre under the direction of M. Prony; by Edward Sang, Dec. 1874.

"Specimen Pages of a Table of the Logarithms of all Numbers up to One Million;" in preparation by Edward Sang.

"The Cone and its Sections treated Geometrically;" by S. A. Renshaw, 1875.

"Bemerkungen zur Theorie der Ternären cubischen Formen;" von Axel Harnack, 1875.

"Fifth Annual Report of Association for the Improvement of Geometrical Teaching," 1875.

"Table des Fonctions Symétriques de Poids xi.," dressée par le Chev. F. Faà de Bruno, Mars 1875.

"Studien über MacLaurin's geometrische Darstellung elliptischer Integrale;" von Dr. Felix Müller, 1875.

"Journal de l'École Polytechnique," 44$^{\text{ieme}}$ cahier, tome xxvii., 1874.

"On the Geometrical Representation of some familiar cases of Reaction in Rigid Dynamics;" by Rev. Prof. Townsend, F.R.S.

"Sui determinanti di Funzioni;" nota del Prof. Felice Casorati.

"Alcune formole fondamentali per lo studio delle equazioni algebrico-differenziali di primo ordine e secondo grado tra due variabili ad integrale generale algebrica;" by the same.

"Tafeln complexer Primzahler welche aus Wurzeln der Einheit gebildet sind;" by C. G. Reuschle, Berlin, 1875.

"Sur la résolution des Équations du 2$^{\text{me}}$, 3$^{\text{me}}$ et 4$^{\text{me}}$ degré par la fonction $r\overline{(X)}$;" by Dr. Axel S. Guldberg.

"Beretning om der internationale Meter-Commissions Mφde i Paris, 24 Sepr., 12 Octbr. 1872;" by Dr. O. J. Broch.

"Partielle Differential-Gleichungen I. O., in denen die unbekannte Funktion explicite vorkommt;" by Sophus Lie.

"Neue Integrations-Methode eines $2n$-gliedrigen Pfaffschen Problems;" by the same.

"Ueber partielle Differential-Gleichungen I. O.;" by the same.

"Zur analytischen Theorie der Berührungs-Transformationen;" by the same.

"Bidrag til Theorien for Dissociationen;" by C. M. Guldberg.

"Bemaerkninger om Formelen for Hφidemaaling med Barometer;" by the same.

"Repertorium der Naturwissenschaften, monatliche Uebersicht der neuesten Arbeiten auf dem Gebiete der Naturwissenschaften," 1 Jahrgang, Nos. 1–6.

"Ueber die Verwerthung der elliptischen Functionen für die Geometrie der Curven dritten Grades," by Axel Harnack, Leipzig, 1875.

"Abstracts and Results of Magnetical and Meteorological Observations at the Magnetic Observatory, Toronto, from 1841–1871," Toronto, 1875.

"Ueber einen Beweis des Abel'schen Theoremes," by Axel Harnack.

"Le Valhalla des Sciences pures et appliquées," by the Count Leopold Hugo, Paris, 1875.

F. W. A. Argelander's "Bibliothek, Astronomie, Mathematik, Physik," Bonn.

"Zur Theorie der Ternären cubischen Formen," by Axel Harnack.

"Zum Andenken an Otto Hesse," by Dr. Klein.

"Almindelige egenskaber ved Systemer af plane Kurver, med anvendelse til Bestemmelse af Karakteristikerne i de elementaere Systemer af fjerde Orden med 5 Tavler af H. G. Zeuthen," avec un résumé en français, Kjφbenhavn, 1873.

Cartes de Visite had been received for the Society's Album from the following gentlemen:—

Prof. Luigi Cremona, Rev. W. H. Laverty, Prof. A. W. Reinold, Messrs. C. J. Monro, J. Walmsley, C. J. C. Price, Col. Clarke, Messrs. W. H. Besant, F.R.S., J. Griffiths, and C. Smith.

Other presents for the Album were a Carte de Visite of the late Dr. Rutherford. and a photographic copy of a likeness of the astronomer Schumacher.

The Meeting proceeded to the election of the new Council. The Scrutators (Messrs. Hammond and Rawson) having examined the balloting lists, declared the following gentlemen (nominated by the Council) duly elected:—

President, Prof. H. J. S. Smith, F.R.S.; Vice-Presidents, Dr. Hirst,

F.R.S.; Lord Rayleigh, F.R.S.; W. Spottiswoode, F.R.S.; Treasurer, S. Roberts, M.A.; Hon. Secretaries, M. Jenkins, M.A., and R. Tucker, M.A. Other Members, Prof. Cayley, F.R.S., Prof. Clifford, F.R.S., T. Cotterill, M.A., J. W. L. Glaisher, F.R.S., Rev. R. Harley, F.R.S, *Prof. Henrici, F.R.S., C. W. Merrifield, F.R.S., Prof. Sylvester, F.R.S., *H. M. Taylor, M.A. (The gentlemen marked * were not on the previous Council list.)

Prof. H. W. Lloyd Tanner, M.A., Jesus College, Oxford, was elected a Member; Major J. R. Campbell, and Prof. G. M. Minchin, M.A., were proposed for election; Messrs. Jeffery and Rhodes were admitted into the Society. Visitors present were Mr. John Ball, Dr. Inglis, Mr. Tait, and the Rev. Alexander Taylor.

Prof. Sylvester gave a detailed account of results arrived at in his paper "On the Fifteen young Ladies Problem, and a General Mathematical Theory of Pure Syntax."

The following further communications were made to the Society:—Mr. J. Hammond, "On the Relation between Bernoulli's Numbers and the Binomial Coefficients;" Mr. S. Roberts, "On Three-bar Motion in Plane Space;" Mr. J. W. L. Glaisher, "Values of Certain Infinite Products, with an Application to the Summation of a Geometrical Series of the n^{th} Order as a Definite Integral;" Major J. R. Campbell, "On the Form of Cam which, acting on a Lever, shall communicate a motion such that the angular velocity-ratio of the lever and cam is a given function of the angle described by the latter" (communicated by the Secretary).

On the Relation between Bernoulli's Numbers and the Binomial Coefficients. By J. Hammond, B.A.

[*Read November 11th*, 1875.]

The expansion of $\frac{\psi(x)}{\phi(x)}$ is

$$\frac{\psi}{\phi} - \left(\frac{1}{\phi}\right)^2 \begin{vmatrix} \psi & \phi \\ \psi' & \phi' \end{vmatrix} x + \left(\frac{1}{\phi}\right)^3 \begin{vmatrix} \psi & \phi & 0 \\ \psi' & \phi' & \phi \\ \psi'' & \phi'' & 2\phi' \end{vmatrix} \frac{x^2}{\underline{2}}$$

$$- \left(\frac{1}{\phi}\right)^4 \begin{vmatrix} \psi & \phi & 0 & 0 \\ \psi' & \phi' & \phi & 0 \\ \psi'' & \phi'' & 2\phi & \phi \\ \psi''' & \phi''' & 3\phi'' & 3\phi' \end{vmatrix} \frac{x^3}{\underline{3}} + \&c.$$

And that of $\frac{1}{\phi(x)}$ is

$$\frac{1}{\phi} - \frac{\phi'}{(\phi)^2}x + \left(\frac{1}{\phi}\right)^3 \begin{vmatrix} \phi' & \phi \\ \phi'' & 2\phi' \end{vmatrix} \frac{x^2}{\lfloor 2} - \left(\frac{1}{\phi}\right)^4 \begin{vmatrix} \phi' & \phi & 0 \\ \phi'' & 2\phi' & \phi \\ \phi''' & 3\phi'' & 3\phi' \end{vmatrix} \frac{x^3}{\lfloor 3} + \&c.$$

(Proceedings, vol. vi., p. 69.)

These give values for Bernoulli's numbers in the form of determinants of the coefficients of the Binomial Theorem.

I. Expand

$$\frac{x}{e^x-1} = \frac{1}{1+\frac{x}{2}+\frac{x^2}{3\lfloor 2}+\&c.} = 1 - \frac{x}{2} + \frac{B_1 x^2}{\lfloor 2} - \frac{B_2 x^4}{\lfloor 4} + \&c.,$$

by means of the second formula, and equate coefficients.

Here $\phi = 1, \quad \phi' = \frac{1}{2}, \quad \phi'' = \frac{1}{3}$ .....

Thus
$$B_n = (-1)^{n-1} \begin{vmatrix} \frac{1}{2}, & 1, & 0, & 0 \ldots \\ \frac{1}{3}, & 1, & 1, & 0 \ldots \\ \frac{1}{4}, & 1, & \frac{3}{2}, & 1 \ldots \\ \frac{1}{5}, & 1, & 2, & 2 \ldots \\ \ldots & \ldots & \ldots & \end{vmatrix} \quad (2n \text{ rows})$$

$$= \frac{(-1)^{n-1}}{\lfloor 2n+1} \begin{vmatrix} 1, & 2, & 0, & 0, & 0, & 0 \ldots \\ 1, & 3, & 3, & 0, & 0, & 0 \ldots \\ 1, & 4, & 6, & 4, & 0, & 0 \ldots \\ 1, & 5, & 10, & 10, & 5, & 0 \ldots \\ 1, & 6, & 15, & 20, & 15, & 6 \ldots \\ 1, & 7, & 21, & 35, & 35, & 21 \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \end{vmatrix} \quad (2n \text{ rows}) \ldots (1).$$

All the numbers in this determinant are coefficients of the Binomial Theorem; for the r^{th} row is

$$\phi^r, \quad r\phi^{r-1}, \quad \frac{r(r-1)}{\lfloor 2}\phi^{r-2} \ldots\ldots, \text{ or } \frac{1}{r+1}, \quad 1, \quad \frac{r}{\lfloor 2} \ldots\ldots,$$

which, when multiplied by $(r+1)$, becomes

$$1, \quad r+1, \quad \frac{(r+1)r}{\lfloor 2} \ldots\ldots$$

II.
$$\frac{e^x-1}{e^x+1} = \frac{e^x+e^{-x}-2}{e^x-e^{-x}} = \frac{\frac{1}{2}x + \frac{1}{4}\frac{x^3}{\lfloor 3} + \frac{1}{6}\frac{x^5}{\lfloor 5} + \&c.}{1 + \frac{1}{3}\frac{x^2}{\lfloor 2} + \frac{1}{5}\frac{x^4}{\lfloor 4} + \&c.}$$

$$= (2^2-1)B_1 x - \frac{2^4-1}{2}B_2\frac{x^3}{\lfloor 3} + \frac{2^6-1}{3}B_3\frac{x^5}{\lfloor 5} - \&c.$$

Expand by means of the first formula and equate the coefficients.

Thus $(-1)^{n-1}\frac{B_n}{n}(2^{2n}-1) = -\begin{vmatrix} 0, & 1, & 0, & 0, & 0, & 0 \ldots \\ \frac{1}{2}, & 0, & 1, & 0, & 0, & 0 \ldots \\ 0, & \frac{1}{3}, & 0, & 1, & 0, & 0 \ldots \\ \frac{1}{4}, & 0, & 1, & 0, & 1, & 0 \ldots \\ 0, & \frac{1}{5}, & 0, & 2, & 0, & 1 \ldots \\ \frac{1}{6}, & 0, & 1, & 0, & \frac{10}{3}, & 0 \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \end{vmatrix}$ ($2n$ rows);

therefore $B_n = \frac{n}{2^{2n}-1}\begin{vmatrix} \frac{1}{2}, & 1, & 0 \ldots \\ \frac{1}{4}, & 1, & 1 \ldots \\ \frac{1}{6}, & 1, & \frac{10}{3} \ldots \\ \ldots & \ldots & \ldots \end{vmatrix}$ (n rows)

$$= \frac{1}{(2^{2n}-1)\,2^n\,\underline{|n-1}}\begin{vmatrix} 1, & 2, & 0 \ldots \\ 1, & 4, & 4 \ldots \\ 1, & 6, & 20 \ldots \\ \ldots & \ldots & \ldots \end{vmatrix} \quad (n \text{ rows}) \ldots (2).$$

The determinant (2) may be obtained from the determinant (1) by striking out all even rows, and all odd columns except the first.

III. $$\tan x = \frac{x - \frac{x^3}{\underline{|3}} + \&c.}{1 - \frac{x^2}{\underline{|2}} + \&c.}$$

$$= \frac{2^2(2^2-1)\,B_1 x}{2} + \frac{2^4(2^4-1)\,B_2}{4}\frac{x^3}{\underline{|3}} + \&c.$$

Expand by the first formula, and equate coefficients.

Thus

$$B_n = -\frac{2n}{2^{2n}(2^{2n}-1)}\begin{vmatrix} 0, & 1, & 0, & 0, & 0, & 0, & 0, & 0 \ldots \\ 1, & 0, & 1, & 0, & 0, & 0, & 0, & 0 \ldots \\ 0, & -1, & 0, & 1, & 0, & 0, & 0, & 0 \ldots \\ -1, & 0, & -3, & 0, & 1, & 0, & 0, & 0 \ldots \\ 0, & 1, & 0, & -6, & 0, & 1, & 0, & 0 \ldots \\ 1, & 0, & 5, & 0, & -10, & 0, & 1, & 0 \ldots \\ 0, & -1, & 0, & 15, & 0, & -15, & 0, & 1 \ldots \\ -1, & 0, & -7, & 0, & 35, & 0, & -21, & 0 \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \end{vmatrix} \quad (2n \text{ rows})$$

$$= \frac{2n(-1)^{n-1}}{2^{2n}(2^{2n}-1)}\begin{vmatrix} 1, & 1, & 0, & 0 \ldots \\ -1, & -3, & 1, & 0 \ldots \\ 1, & 5, & -10, & 1 \ldots \\ -1, & -7, & 35, & -21 \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix} \quad (n \text{ rows})$$

$$= \frac{2n}{2^{2n}(2^{2n}-1)} \begin{vmatrix} 1, & 1, & 0, & 0 \ldots \\ 1, & 3, & 1, & 0 \ldots \\ 1, & 5, & 10, & 1 \ldots \\ 1, & 7, & 35, & 21 \ldots \\ \ldots & & \ldots & \ldots \end{vmatrix} \quad (n \text{ rows}) \ldots\ldots\ldots\ldots (3).$$

The determinant (3) may be obtained from the determinant (1) by striking out all odd rows and all odd columns, and bordering the determinant so formed with a border of units both vertically and diagonally.

IV. $$x \operatorname{cosec} x = \frac{1}{1 - \frac{x^2}{3\lfloor 2} + \frac{x^4}{5\lfloor 4} - \&c.}$$

$$= 1 + 2(2-1)\frac{B_1 x^2}{\lfloor 2} + 2(2^3-1)\frac{B_2 x^4}{\lfloor 4} + \&c.$$

Expand by means of the second formula, and equate coefficients.

Thus $$B_n = \frac{1}{2(2^{2n-1}-1)} \begin{vmatrix} 0, & 1, & 0, & 0, & 0, & 0 \ldots \\ -\frac{1}{3}, & 0, & 1, & 0, & 0, & 0 \ldots \\ 0, & -1, & 0, & 1, & 0, & 0 \ldots \\ \frac{1}{5}, & 0, & -2, & 0, & 1, & 0 \ldots \\ 0, & 1, & 0, & -\frac{10}{3}, & 0, & 1 \ldots \\ -\frac{1}{7}, & 0, & 3, & 0, & -5, & 0 \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix} \quad (2n \text{ rows})$$

$$= \frac{1}{2(2^{2n-1}-1)} \begin{vmatrix} \frac{1}{3}, & 1, & 0 \ldots \\ \frac{1}{5}, & 2, & 1 \ldots \\ \frac{1}{7}, & 3, & 5 \ldots \\ \ldots & \ldots & \ldots \end{vmatrix} \quad (n \text{ rows}),$$

which reduces to

$$B_n = \frac{2^{n-1}\lfloor n}{\lfloor 2n+1\,(2^{2n-1}-1)} \begin{vmatrix} 1, & 3, & 0 \ldots \\ 1, & 10, & 5 \ldots \\ 1, & 21, & 35 \ldots \\ \ldots\ldots\ldots & & \end{vmatrix} \quad (n \text{ rows}) \ldots\ldots\ldots\ldots (4).$$

The determinant (4) may be obtained from the determinant (1) by striking out all odd rows, and all even columns.

V. The coefficients of the expansion of sec x may also be expressed as determinants of selected coefficients of the Binomial Theorem.

For $$\sec x = \frac{1}{1 - \frac{x^2}{\lfloor 2} + \frac{x}{\lfloor 4} - \&c.} = 1 + \frac{E_2}{\lfloor 2} x^2 + \frac{E_4}{\lfloor 4} x^4 + \ldots\ldots$$

gives

$$E_{2n} = \begin{vmatrix} 0, & 1, & 0, & 0, & 0, & 0, & 0, & 0 \ldots \\ -1, & 0, & 1, & 0, & 0, & 0, & 0, & 0 \ldots \\ 0, & -3, & 0, & 1, & 0, & 0, & 0, & 0 \ldots \\ 1, & 0, & -6, & 0, & 1, & 0, & 0, & 0 \ldots \\ 0, & 5, & 0, & -10, & 0, & 1, & 0, & 0 \ldots \\ -1, & 0, & 15, & 0, & -15, & 0, & 1, & 0 \ldots \\ 0, & -7, & 0, & 35, & 0, & -21, & 0, & 1 \ldots \\ 1, & 0, & -28, & 0, & 70, & 0, & -28, & 0 \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \end{vmatrix} \quad (2n \text{ rows}).$$

This reduces to $E_{2n} = \begin{vmatrix} 1, & 1, & 0, & 0 \ldots \\ 1, & 6, & 1, & 0 \ldots \\ 1, & 15, & 15, & 1 \ldots \\ 1, & 28, & 70, & 28 \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix}$ (n rows) ..... (5).

The determinant (5) may be obtained from the determinant (1) by striking out all even rows and all even columns, and bordering the determinant so formed with a border of units diagonally.

VI. To verify the truth of these formulæ, put $n=1, 2, 3$ in them.

Then, when $n=1$, each of the first four gives $B_1 = \frac{1}{6}$.

When $n=2$, the first four formulæ give

(1) ..... $B_2 = -\frac{1}{120}\begin{vmatrix} 1, & 2, & 0, & 0 \\ 1, & 3, & 3, & 0 \\ 1, & 4, & 6, & 4 \\ 1, & 5, & 10, & 10 \end{vmatrix} = -\frac{1}{120}(1.0 - 2.2) = \frac{1}{30}.$

(2) ..... $B_2 = \frac{1}{15.4.1}\begin{vmatrix} 1, & 2 \\ 1, & 4 \end{vmatrix} = \frac{1}{30}.$

(3) ..... $B_2 = \frac{4}{16.15}\begin{vmatrix} 1, & 1 \\ 1, & 3 \end{vmatrix} = \frac{1}{30}.$

(4) ..... $B_2 = \frac{2.2}{120.7}\begin{vmatrix} 1, & 3 \\ 1, & 10 \end{vmatrix} = \frac{1}{30}.$

When $n = 3$,

(1) ..... $B_3 = \frac{1}{\lfloor 7}\begin{vmatrix} 1, & 2, & 0, & 0, & 0, & 0 \\ 1, & 3, & 3, & 0, & 0, & 0 \\ 1, & 4, & 6, & 4, & 0, & 0 \\ 1, & 5, & 10, & 10, & 5, & 0 \\ 1, & 6, & 15, & 20, & 15, & 6 \\ 1, & 7, & 21, & 35, & 35, & 21 \end{vmatrix} = \frac{1}{\lfloor 7}\begin{vmatrix} 1, & 2, & 0, & 0, & 0, & 0 \\ 0, & 1, & 3, & 0, & 0, & 0 \\ 0, & 1, & 3, & 4, & 0, & 0 \\ 0, & 1, & 4, & 6, & 5, & 0 \\ 0, & 1, & 5, & 10, & 10, & 6 \\ 0, & 1, & 6, & 15, & 20, & 15 \end{vmatrix}$

$$= \frac{1}{\lfloor 7} \begin{vmatrix} 1, & 2, & 0, & 0, & 0, & 0 \\ 0, & 1, & 3, & 0, & 0, & 0 \\ 0, & 0, & 0, & 4, & 0, & 0 \\ 0, & 0, & 1, & 2, & 5, & 0 \\ 0, & 0, & 1, & 4, & 5, & 6 \\ 0, & 0, & 1, & 5, & 10, & 9 \end{vmatrix} = \frac{1}{\lfloor 7} \begin{vmatrix} 1, & 2, & 0, & 0, & 0, & 0 \\ 0, & 1, & 3, & 0, & 0, & 0 \\ 0, & 0, & 0, & 4, & 0, & 0 \\ 0, & 0, & 1, & 2, & 5, & 0 \\ 0, & 0, & 0, & 2, & 0, & 6 \\ 0, & 0, & 0, & 1, & 5, & 3 \end{vmatrix}$$

$$= \frac{4 \cdot 5 \cdot 6}{\lfloor 7} = \frac{1}{42}.$$

$$(2) \ldots\ldots B_3 = \frac{1}{63 \cdot 8 \cdot 2} \begin{vmatrix} 1, & 2, & 0 \\ 1, & 4, & 4 \\ 1, & 6, & 20 \end{vmatrix} = \frac{56 - 2 \cdot 16}{63 \cdot 8 \cdot 2} = \frac{1}{42}.$$

$$(3) \ldots\ldots B_3 = \frac{6}{64 \cdot 63} \begin{vmatrix} 1, & 1, & 0 \\ 1, & 3, & 1 \\ 1, & 5, & 10 \end{vmatrix} = \frac{(25-9)\,6}{64 \cdot 63} = \frac{1}{42}.$$

$$(4) \ldots\ldots B_3 = \frac{4 \cdot 6}{\lfloor 7\ (32-1)} \begin{vmatrix} 1, & 3, & 0 \\ 1, & 10, & 5 \\ 1, & 21, & 35 \end{vmatrix} = \frac{4 \cdot 6\,(350 - 105 - 3 \cdot 30)}{31 \lfloor 7}$$

$$= \frac{4 \cdot 6 \cdot 155}{31 \lfloor 7} = \frac{1}{42}.$$

The formula (5) gives

$$\sec x = 1 + \frac{x^2}{\lfloor 2} + \begin{vmatrix} 1, & 1 \\ 1, & 6 \end{vmatrix} \frac{x^4}{\lfloor 4} + \begin{vmatrix} 1, & 1, & 0 \\ 1, & 6, & 1 \\ 1, & 15, & 15 \end{vmatrix} \frac{x^6}{\lfloor 6} + \ldots\ldots$$

Or

$$\sec x = 1 + \frac{x^2}{\lfloor 2} + \frac{5x^4}{\lfloor 4} + \frac{61x^6}{\lfloor 6} + \ldots\ldots$$

This result may be verified by ordinary division, or otherwise.

On Three-bar Motion in Plane Space. *By* S. Roberts, M.A.

[*Read Nov. 11th*, 1875.]

1. I propose to extend briefly to general three-bar motion a discussion which I have before given of some particular cases; and fortunately I find my task rendered easy by some recent discoveries in relation to the subject.

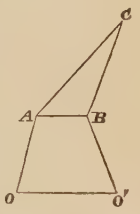
It is known that the locus of a point rigidly attached to a link connected immediately with two others moving in a plane about two

fixed points, is a sextic tricircular curve having three nodes besides the triple circular points.

If we consider the general movement as that of a plane moving with two of its points on two fixed circles in the same level, there are nine constants at our disposal; namely, the three parameters of each of the circles and the three parameters connecting the describing point with the points conditioned to move in circles.

Practically, and with reference to what is more strictly regarded as mechanical three-bar motion, there are further limitations, since the bars are finite. Nevertheless, it will be convenient to consider generally the curves represented by the analytical equation. In the first instance, therefore, I give one form out of many of the equation of the locus.

2. Our curve is generated by the vertex C of a triangle ABC, moving with its base angles A and B on two circles, having respectively their centres at O, O′, and the radii OA and O′B.



Take O as the origin of rectangular axes and let OO′ be the axis of x. Let the co-ordinates of A, B, C respectively be (x', y'), (x'', y''), and (x, y).

Then, denoting as usual the sides and angles of the triangle by a, b, c, A, B, C, and by θ the angle which AC makes with the axis of x, we have

$$x' = x - b\cos\theta, \quad x'' = x - a\cos(C+\theta),$$
$$y' = y - b\sin\theta, \quad y'' = y - a\sin(C+\theta).$$

and writing r for OA, and s for O′B, and γ for OO′, we have also

$$x'^2 + y'^2 = r^2, \quad (x''-\gamma)^2 + y''^2 = s^2.$$

Determining therefore $\cos\theta$, $\sin\theta$ by means of

$$x\cos\theta + y\sin\theta = \frac{x^2+y^2+b^2-r^2}{2b},$$

$$\{(x-\gamma)\cos C + y\sin C\}\cos\theta - \{(x-\gamma)\sin C - y\cos C\}\sin\theta = \frac{(x-\gamma)^2+y^2+a^2-s^2}{2a};$$

and putting k for $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$, we get for the equation of the locus

$$\left\{\begin{matrix}\sin A\,[(x-\gamma)\sin C - y\cos C]\,[x^2+y^2+b^2-r^2] \\ +\sin B y\,[(x-\gamma)^2+y^2+a^2-s^2]\end{matrix}\right\}^2$$
$$+\left\{\begin{matrix}\sin A\,[(x-\gamma)\cos C + y\sin C]\,[x^2+y^2+b^2-r^2] \\ -\sin B x\,[(x-\gamma)^2+y^2+a^2-s^2]\end{matrix}\right\}^2$$
$$= 4k^2\sin^2 A\sin^2 B\sin^2 C\,[x(x-\gamma)+y^2-\gamma\cot C y]^2 = 0 \quad \ldots(1).$$

3. The equation is of the form

$$(L\phi+M\psi)^2+(N\phi+P\psi)^2 = K\,(LP-MN)^2,$$

where ϕ, ψ, and $LP-MN$ represent circles; and it is immediately seen that the three finite nodes lie on the last circle.

It can be shown geometrically, and very simply, that OO′ subtends an angle C, or $180°-C$, at a node. For, suppose ABC, A′B′C are the two positions of the traversing triangle which determine a node; then we have

$$\angle OCO' = \angle OCA - \angle O'CB + \angle ACB = C,$$

or

$$\angle OCO' = \angle OCA + \angle O'CB - \angle ACB = 180°-C.$$

Again, if ρ, ρ' are the distances of the node at C from O, O′ respectively, we have

$$r^2 = \rho^2+b^2-2\rho b\cos A'CO,$$
$$s^2 = \rho'^2+a^2-2\rho a\cos A'CO,$$
$$\gamma^2 = \rho^2+\rho'^2-2\rho\rho'\cos C;$$

giving, to determine the nodes,

$$\rho^2\{\sin C\cos(C-B)\,\rho^2+\sin B\,(\gamma^2+a^2-s^2)+\sin A\cos C\,(b^2-r^2)\}^2$$
$$-(\gamma^2-\sin^2 C\rho^2)\{\sin(C-B)\,\rho^2+\sin A\,(b^2-r^2)\}^2 = 0 \ldots\ldots (2).$$

4. I remark, also, that the two centres O, O′ are singular foci ("foyers singuliers" of M. Laguerre), that is to say, real intersections of the circular asymptotes. There is plainly another such focus, and it is natural to infer that it lies on the circle $LP-MN=0$, as do the two foci O and O′.

In fact, by substituting X for $\frac{x+iy}{2}$ and Y for $\frac{x-iy}{2}$ in (1), and taking the coefficient of Y^3, we get the tangents at one of the circular points, and they are found to be

$$x+iy=0,\quad x-\gamma+iy=0,\quad x-\gamma\frac{\sin B}{\sin C}\cos A+i\left(y-\gamma\frac{\sin A\sin B}{\sin C}\right).$$

Hence the chord OO′ subtends the angle C at the third focus, and we see that the triangle of foci is similar to the traversing triangle.

We also see, at once, how it is impossible to describe by these mechanical means curves having coincident tangents at the circular points, *e. g.*, the Limaçon of Pascal; for, in that case, two or more of the foci would coincide. The foci do indeed coincide when the directrix-circles are concentric, and the locus degenerates.

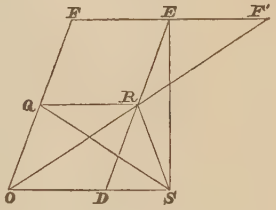
Since the absolute term of (2) is $-\frac{\sin^2 A\gamma^2(b^2-r^2)^2}{\sin^2 C}$, if $b^2=r^2$, the focus at the origin is also a node; if also $a^2=s^2$, the point O′ is a node; and in this case the third focus is also a node. If $a=b=\epsilon$, we have a remarkably symmetrical curve described in three ways by the same linkage with different pairs of centres.

5. We are thus led to remark that, since the singular foci are similarly related to the locus, we might have taken as fixed centres the focus O, and the third focus, say O″, and by means of links of suitable lengths we should obtain the same locus. And in like manner we might have taken as centres O′ and O″. Moreover, the triangle of foci may be said to be not only similar to the traversing triangle, but similarly disposed with regard to the centres about which the traversing triangle rotates. For the angle A of the triangle of foci is at O the centre of the circle on which the angle A of the traversing triangle moves, and the angle B of the triangle of foci is at O′.

We conclude, then, that the locus in question can be described in three different ways, by similar traversing triangles moving with an angle on each of two circles about two singular foci as centres, and the angle moving on a circle is equal to the angle subtended at its centre by the two remaining singular foci.

6. I have already had occasion to notice a particular case of the foregoing threefold generation, namely, when the locus breaks up into a circle and the pedal of a conic.

Thus, let F, F′, and E be the foci and centre of the conic of which part of the locus is the pedal. Then, if Q, R, S are the singular foci of the complete locus, the last being the centre of the extraneous circle, I have shewn that, O being the node of the pedal,

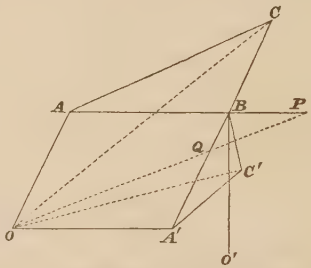


$$\angle \mathrm{ROS} = \angle \mathrm{QSO},$$
$$\angle \mathrm{QOS} = \angle \mathrm{RSO}.$$

Also, if Q, S are taken as centres of the directrix circles, the radius of the circle about Q as centre is $\frac{\mathrm{OF}'}{2}$, and the radius of the circle about S as centre is $\frac{\mathrm{OF}'}{2e}$, where e is the eccentricity of the conic. Similarly, if we take R and S as centres, the radius of the circle about R as centre is $\frac{\mathrm{OF}}{2}$, and that of the circle about S as centre is $\frac{\mathrm{OF}}{2e}$. If, again, we take circles about Q and R as centres for directrices, their radii are equal to semi-major axis of the conic. The traversing triangle is similar, in each case, to the triangle QRS, and similarly disposed in the sense previously mentioned. In the second mode of generation, the node O is determined as at a distance OD = FE from the extremity D of a diameter of the circle about R as centre, where it intersects OS, which is parallel to QR and FF′. This is the determination of the node given in effect in my paper on the Pedals of Conics (*Proc.*

No. 26). M. Mannheim has found it in a different way (*Proc.*, No. 77, p. 35), where the figure gives a hyperbola as primitive, while the above implies an ellipse.

7. It remains to determine the relations existing among the radii of the different directrix circles in the general case. Fortunately we can avoid the tedious process of deriving these from the equation by making use of Dr. Sylvester's discovery of the "Plagiograph," or Skew Pantograph, which is, in fact, the key to the theory of three-bar motion.



In Fig. 3, OA, O′B are the rotating links, AB is the traversing link, and ABC is the traversing triangle. Then, completing the parallelogram OA′BA, and taking AP in AB = AC, draw OP meeting A′B in Q. Make $\angle BA'C' = \angle BAC$, and $A'C' = A'Q$; then C′ describes a curve similar to that described by C, but twisted through an angle A.

Now, the triangle A′BC′ is similar to the triangle ABC, but differently placed with regard to the traversing link A′B, viz., $\angle BA'C' = \angle A$, $\angle A'C'B = \angle B$, $\angle A'BC' = \angle C$. In order to make the transformed figure equal as well as similar to the primitive, we must alter its magnitude in the ratio $\frac{AC}{AB}$. Hence we see at once that, if OA_1 is the radius of the other directrix circle with centre O, we must have $OA_1 = AC$. That is to say, the radius of a directrix circle about O as centre is equal to the adjacent side of the traversing triangle, when the conjugate circle about the same centre is employed.

In order better to realise the result, suppose that the three singular foci form a triangle OO_1O_2, and that ABC, $A_1B_1C_1$, $A_2B_2C_2$ are the traversing triangles corresponding to the centres (O_1, O_2) (O, O_2), (O, O_1) respectively.

Then we have, using the usual notation,

$$O_1A = c_2, \qquad O_2B_1 = a, \qquad OC_1 = a_2,$$
$$O_1A_2 = b, \qquad O_2B = c_1, \qquad OC_2 = b_1.$$

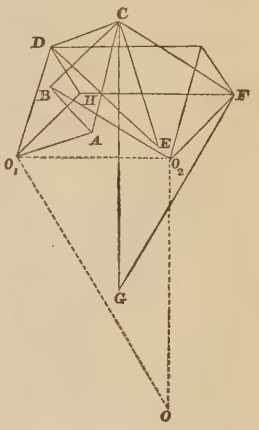
Hence $$OC_1 \cdot O_1A_2 \cdot O_2B = OC_2 \cdot O_1A \cdot O_2B_1;$$

and, in terms of a, b, c, $O_1A = r$, and $O_2B = s$, the sides of the three triangles are (a, b, c), $\left(\frac{a}{c}s, \frac{b}{c}s, s\right)$, $\left(\frac{a}{c}r, \frac{b}{c}r, r\right)$.

8. We are now in a position to draw for any point of the locus the corresponding positions of the traversing triangles, and determine a remarkable linkage, of which I think it is worth while to give a figure, omitting some interesting connections, which can be easily supplied.

I observed in a paper "On the Motion of a Plane under certain conditions,"* that, from the form of the analytical equations for obtaining the locus of a point rigidly connected with a limited line moving with its extremities on two given curve directrices, the curve so generated was also generated by the intersections of the two curves supposed to move always parallel to themselves, and translated to the extremities of two links, inclined at a constant angle, and rotating about two points.

That is to say, in the case we are dealing with, O_1, O_2 being the fixed centres, O_1O_2FH is a parallel ruler; D is a point rigidly connected with O_1H; DC, CF are two links moving about D, C, and F; then C describes our locus. The thing is evident immediately we complete the parallelograms O_1DCA, O_2FCB, and join AB; for the triangle ABC is equal to the constant triangle O_1DH. Dr. Sylvester's Skew Pantograph can now be applied; that is to say, making $\angle EDC = \angle BAC$, $\angle DCE = \angle ABC$, we get another constant triangle DCE, whose vertex E moves on a circle about O as centre, O_1O_2O being the triangle of foci.



Similarly we determine the triangle CFG, moving with G on a circle about O as centre.

If the figure were completed, we should have a system of nine triangular links, all similar and equal in threes, a pair rotating about each of the foci O, O_1, O_2, and the other three joining at the describing point and connected with the rotating links in the manner above indicated.

Many of the relations of this linkage are obvious on inspection, and I need not dwell on them. It will be seen that if we take two of the foci and the two links rotating about each of them, the two links which do not belong to the three-bar linkage remain inclined at a constant angle, and their intersection, when prolonged, lies on the circle of nodes and foci. The intersection of the others gives the instantaneous centre. Thus for the system we get three such centres, which lie on and determine the normal. The locus of instantaneous centres for each three-bar movement is a curve of the eighth order, having quadruple points at the two corresponding singular foci, passing twice through the circular points, and having the deficiency unity. The corresponding loci on the moving planes are of the same kind. When the describing point is on the traversing link, the foregoing constructions are, of course, much simplified. The rotating links are parallel in pairs, and the line joining two extremities of parallel links passes through the remaining focus, as follows from similar triangles.

* *Proc.*, Vol. III., p. 287.

9. The equation (1) has been written in a form which shows at once that, if b^2-r^2, a^2-s^2 remain constant, and the foci are given, the curves form a faisceau of sextics with the parameter k. The fundamental points are the nodes and triple circular points, multiple in all the curves, and the points of contact of the tangents at the triple points.

For $\frac{1}{k}=0$, the locus is a circle twice and the line at infinity twice. The traversing triangle is infinite, but the describing angle is C, and in effect the rotating links OA, O′B prolonged constitute two sides of the triangle, and C describes the circle of nodes and foci.

Again, if $k=0$, we have two imaginary cubics, each having a node at a circular point, and passing through the remaining one, the nodes being different in the two curves, the tangents at the nodes and the other circular points constitute the six circular asymptotes of the sextic, and give the singular foci.

It can now be shown that the family of sextics in question is defined by the conditions—(1) the circular points triple; (2) three other nodes and the real singular foci lie on a circle; (3) the triple points and nodes, and the six points of contact of tangents at the triple points, form a complete intersection of sextics.*

For, as one fundamental curve of such a faisceau we may take a doubled circle and the line at infinity twice, since this combination fulfils the analytical conditions. And again, if we write X for $\frac{x+iy}{2}$, Y for $\frac{x-iy}{2}$, and Z for unity, so that

$$tY(X+aZ)-sX(Y+aZ)=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (a)$$

is a circle, and $XY(X+aZ)+bX^2Z+cXZ^2+dZ^3=0$ $\ldots\ldots\ldots\ldots (b)$

is a cubic having a node at XZ, with tangents X and X+aZ, and a simple point at YZ. We may determine b, c, d so that the cubic passes through three arbitrary points on the circle (a). Suppose this to be done. We may next multiply (b) by $\frac{a}{a-b}$ or $1+\frac{b}{a-b}$, and write it in the form

$$(X+aZ)\left(XY+\frac{d}{a-b}Z^2\right)+\frac{bX}{a-b}\left\{XY+a(X+Y)Z+\frac{ac-d}{b}Z^2\right\}=0$$

$\ldots\ldots\ldots (b')$,

* In a paper containing further developments, which will shortly appear, Professor Cayley has stated more directly and simply the connection between the nodes and foci thus: (1) with regard to the centre and a diameter of the circle, the sum of the angular distances of the nodes is equal to the corresponding sum for the foci; (2) otherwise, the triangles of nodes and of foci circumscribe a parabola.

and then the cubic

$$(Y+aZ)\left(XY+\frac{d}{a-b}Z^2\right)+\frac{tb}{s(a-b)}\left\{XY+a(X+Y)Z+\frac{ac-d}{b}Z^2\right\}=0 \quad \ldots\ldots\ldots(c')$$

will pass through the same three finite points on the circle (a), and has a node at YZ and a simple point at XZ, the tangents at the node being $Y=0$, $Y+aZ=0$, which intersect the corresponding tangents to the other cubic upon the given circle.

The cubics are now determined, but it will be observed that the tangents to (b') at YZ, and to (c') at XZ, namely,

$$Y+\frac{b}{a-b}(Y+aZ)=0,$$

$$X+\frac{tb}{s(a-b)}(X+aZ)=0,$$

also intersect on the circle.

Hence, to determine the faisceau, we can always get the form for three-bar motion.

10. In conclusion, I refer shortly to the inverse curves.

The inverse of the general locus with respect to an arbitrary point is also a sextic curve, having the circular points at infinity for triple points and three other nodes. The singular foci, however, do not correspond as inverses to those of the primitive, and the inverse curve is not in general a case of three-bar motion. If the describing point is on the traversing link, the circle of nodes and foci degenerates to a line, or rather breaks up into a finite line, and the line at infinity and the locus is symmetrical with regard to the finite line. The inverse with respect to a point on the axis is also symmetrical with regard to it, and the nodes and singular foci lie on that line. If, then, we take the inverse with respect to (1) any point on the axis, or (2) a node, it is the same family as the primitive, and included analytically as a case of three-bar motion.

I shall verify (1) by identifying the form of the inverse and that of the primitive.

(i.) Multiply equation (1) by k^2, and write a, b for k sin A, k sin B respectively; then, making $C=0$, we get for the equation of the locus, when the describing point is on the link,

$$y^2\{a(x^2+y^2+b^2-r^2)-b[(x-\gamma)^2+y^2+a^2-s^2]\}^2$$
$$+\{a(x^2+y^2+b^2-r^2)-bx[(x-\gamma)^2+y^2+a^2-s^2]\}^2-4a^2b^2\gamma^2y^2=0,$$

which, if ρ^2 is written for x^2+y^2, is of the form

$$y^2(\rho^2+2Px+Q)^2+[x(\rho^2+2Px+Q)+R\rho^2+S]^2+Ty^2=0\ldots\ldots(3),$$

where P, Q, R, S, T are independent constants.

For a new origin on the axis, the general form with six constants is

$$(Ax\rho^2+B\rho^2+Cx+D)^2+y^2(A^2\rho^4+E^2\rho^2+F^2x^2+G^2)=0.$$

The inverse with regard to the new origin, the modulus being unity, is

$$(Ax+B\rho^2+Cx\rho^2+D\rho^4)^2+y^2(A^2+E^2\rho^2+F^2x\rho^2+G^2\rho^4)=0;$$

or, dividing out ρ^2, we get

$$(A+Bx+C\rho^2+Dx\rho^2)^2+y^2\left\{\begin{matrix}D^2\rho^4+(G^2-C^2+2BD)\rho^2+F^2x\\+E^2+B^2-2AC\end{matrix}\right\}=0,$$

the same form as that of the primitive.

(ii.) If we take as origin a node of the symmetrical locus, the inverse is a quartic curve, symmetrical with regard to the same axis. It is binodal and circular, having its nodes on the axis.

I have elsewhere drawn attention to the interesting class of quartics described by a point rigidly connected with a limited line moving with its extremities on a circle and straight line. In the general case the locus is binodal and circular, having its singular focus collinear with the nodes, and having two parallel double tangents, whose points of contact lie on a conic through the nodes.

If the describing point is on the traversing link, we have a family of quartics, symmetrical, circular, and having two nodes on the axis.

The analytical equation includes all such quartics.

For this equation is easily found in the form

$$[a(x^2+y^2+b^2-r^2)-2b(x-k)x]^2-4b^2y^2[a^2-(x-k)^2]=0 \quad \ldots(4),$$

the centre of the circle directrix being the origin, the equation of the rule line being $x-k=0$, and a, b being the distances of the describing point from the circle and line respectively. Now, if the singular focus is the origin, a symmetrical circular quartic with two nodes on the axis is represented by

$$y^2[y^2+(1+P^2)x^2+2PQx+R]+(Px^2+Qx+S)^2=0,$$

or $$[y^2+Px^2+Qx+S]^2+(1-P)^2x^2+2(P-1)Qxy^2+(R-2S)y^2=0,$$

or $$\left[x^2+y^2+S-(1-P)x\left(x-\frac{Q}{1-P}\right)\right]^2$$

$$-(1-P)^2y^2\left[\frac{2S-R-Q}{(1-P)^2}-\left(x-\frac{Q}{1-P}\right)^2\right]=0,$$

which is of the form (4).

A line directrix is, of course, a limiting case of a circle directrix. Accordingly, there are two ways of describing the same quartic, the third failing by the traversing triangle becoming infinite.

The foregoing result is interesting as shewing that a parallel motion may be established by adding an inverter and two other links to a symmetrical three-bar linkage. The construction, however, is clearly not always practicable.

Since writing the foregoing I have observed that a class of curves of

three-bar motion, not capable of being so described in a finite manner, are yet given by the following construction:—

If a constant angle moves with a fixed point of one side on a circle, while the other side touches another circle, the locus of the apex is a sextic curve with three finite nodes. The circular points are triple, and at each two tangents are coincident. The two singular foci lie on the circle of nodes. Thus several modes of mechanical description, seemingly far removed from each other, are closely associated, since, analytically, the curves described are included in the equation of three-bar motion curves. I cannot do more here than allude to this result. (See *Proc.*, Vol. II., § 10.)

Values of Certain Infinite Products, with an application to the Summation of the Geometrical Series of the n^{th} Order as a Definite Integral. By J. W. L. Glaisher, M.A., F.R.S.

[*Read November 11th*, 1875.]

Starting with the identity

$$1+x^n = 1-\omega x \,.\, 1-\omega^3 x \,\ldots\, 1-\omega^{2n-1}x,$$

in which $\omega = \cos\frac{\pi}{n} + i\sin\frac{\pi}{n}$ (or any other prime $2n^{\text{th}}$ root of unity), and writing for x successively x^2, $(\frac{1}{2}x)^2$, $(\frac{1}{3}x)^2$, &c., we have

$$1+x^{2n} = 1-(\zeta x)^2 \,.\, 1-(\zeta^3 x)^2 \,\ldots\, 1-(\zeta^{2n-1}x)^2,$$

$$1+\left(\frac{x}{2}\right)^{2n} = 1-\left(\frac{\zeta x}{2}\right)^2 \,.\, 1-\left(\frac{\zeta^3 x}{2}\right)^2 \,\ldots\, 1-\left(\frac{\zeta^{2n-1}x}{2}\right)^2,$$

$$\ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots$$

and

$$x^n = (-)^{\frac{1}{2}n} \,.\, \zeta x \,.\, \zeta^3 x \,\ldots\, \zeta^{2n-1}x,$$

ζ being $= \sqrt{\omega} = \cos\frac{\pi}{2n} + i\sin\frac{\pi}{2n}$;

whence

$$x^n \,.\, 1+x^n \,.\, 1+\frac{x^{2n}}{2^{2n}} \,.\, 1+\frac{x^{2n}}{3^{2n}} \,\ldots = (-)^{\frac{1}{2}n}\frac{1}{\pi^n}\sin\pi\zeta x \sin\pi\zeta^3 x \,\ldots\, \sin\pi\zeta^{2n-1}x$$

$$= (-)^{\frac{1}{2}n}\frac{1}{\pi^n}\sin\pi(\beta_1+i\alpha_1)x \sin\pi(\beta_3+i\alpha_3)x \,\ldots\, \sin\pi(\beta_{2n-1}+i\alpha_{2n-1})x;$$

whence, pairing the first and last factors, the second and the last but one, &c., and observing that

$$\sin(a+bi)\sin(-a+bi) = -\tfrac{1}{2}(\cosh 2b - \cos 2a)$$

and that, if n be uneven, the middle factor $\sin\pi\zeta^n x = i\sinh\pi x$, we obtain—

I. $$x^n \,.\, 1+x^{2n} \,.\, 1+\frac{x^{2n}}{2^{2n}} \,.\, 1+\frac{x^{2n}}{3^{2n}} \,\ldots = \frac{1}{2^{\frac{1}{2}n}\pi^n}(\cosh 2\pi\alpha_1 x - \cos 2\pi\beta_1 x)$$
$$\times(\cosh 2\pi\alpha_3 x - \cos 2\pi\beta_3 x) \,\ldots\, (\cosh 2\pi\alpha_{n-1}x - \cos 2\pi\beta_{n-1}x);$$

or, say, $= \frac{1}{2^{\frac{1}{2}n}\pi^n}\Pi_1^{n-1}(\cosh 2\pi\alpha x - \cos 2\pi\beta x)$, if n be even;

and $= \frac{1}{2^{\frac{1}{2}(n-1)}\pi^n}\sinh \pi x \Pi_1^{n-2}(\cosh 2\pi\alpha x - \cos 2\pi\beta x)$,
if n be uneven;

where $\alpha_r = \sin\frac{r\pi}{2n}, \quad \beta_r = \cos\frac{r\pi}{2n}$

(so that $\alpha_r = \beta_{n-r}$).

II. From these formulæ we derive

$$1+x^{2n}.1+\frac{x^{2n}}{3^{2n}}.1+\frac{x^{2n}}{5^{2n}}\ldots = \frac{1}{2^{\frac{1}{2}n}}\Pi_1^{n-1}(\cosh \pi\alpha x + \cos \pi\beta x),$$ if n be even;

and $= \frac{1}{2^{\frac{1}{2}(n-1)}}\cosh \tfrac{1}{2}\pi x \Pi_1^{n-2}(\cosh \pi\alpha x + \cos \pi\beta x)$,
if n be uneven.

III. We have also

$$\frac{2^{2n}+x^{2n}.4^{2n}+x^{2n}.6^{2n}+x^{2n}\ldots}{1^{2n}+x^{2n}.3^{2n}+x^{2n}.5^{2n}+x^{2n}\ldots} = \frac{1}{x^n}\Pi_1^{n-1}\frac{\cosh \pi\alpha x - \cos \pi\beta x}{\cosh \pi\alpha x + \cos \pi\beta x},$$ if n be even;

and $= \frac{1}{x^n}\tanh \tfrac{1}{2}\pi x \Pi_1^{n-2}\frac{\cosh \pi\alpha x - \cos \pi\beta x}{\cosh \pi\alpha x + \cos \pi\beta x}$,
if n be uneven;

where in the expression on the left-hand side it is supposed that there is one more factor in the denominator than in the numerator, and that the additional factor is under the square root sign. We thus have a generalization of Wallis's formula for $\frac{1}{2}\pi$.

IV. By differentiating the results in I. and II., it follows that

$$\frac{1}{1^{2n}+x^{2n}}+\frac{1}{2^{2n}+x^{2n}}+\frac{1}{3^{2n}+x^{2n}}+\text{ &c.}$$

$$= \frac{\pi}{nx^{2n-1}}\Sigma_1^{n-1}\frac{\alpha\sinh 2\pi\alpha x + \beta\sin 2\pi\beta x}{\cosh 2\pi\alpha x - \cos 2\pi\beta x} - \frac{1}{2x^{2n}},$$ if n be even;

and $= \frac{\pi}{2nx^{2n-1}}\coth \pi x + \frac{\pi}{nx^{2n-1}}\Sigma_1^{n-2}\frac{\alpha\sinh 2\pi\alpha x + \beta\sin 2\pi\beta x}{\cosh 2\pi\alpha x - \cos 2\pi\beta x} - \frac{1}{2x^{2n}}$,
if n be uneven.

$$\frac{1}{1^{2n}+x^{2n}}+\frac{1}{3^{2n}+x^{2n}}+\frac{1}{5^{2n}+x^{2n}}+\text{ &c.}$$

$$= \frac{\pi}{2nx^{2n-1}}\Sigma_1^{n-1}\frac{\alpha\sinh \pi\alpha x - \beta\sin \pi\beta x}{\cosh \pi\alpha x + \cos \pi\beta x},$$ if n be even;

and $= \frac{\pi}{4nx^{2n-1}}\tanh \tfrac{1}{2}\pi x + \frac{\pi}{2nx^{n-1}}\Sigma_1^{n-2}\frac{\alpha\sinh \pi\alpha x - \beta\sin \pi\beta x}{\cosh \pi\alpha x + \cos \pi\beta x}$,
if n be uneven

(Σ of course, here and throughout, only including uneven value sof the suffixes).

V. From the integral $e^{-ab} = \frac{2}{\pi}\int_0^\infty \frac{x \sin ax}{b^2+x^2}\,dx$,

we derive $q^{r^n} = \frac{2n}{\pi}\int_0^\infty \frac{x^{2n-1}\sin(ax^n)}{r^{2n}+x^{2n}}\,dx, \quad \left(a = \log\frac{1}{q}\right)$;

whence $\frac{1}{2}+q^{1^n}+q^{2^n}+q^{3^n}+\&c.$

$$= \frac{1}{\pi}\int_0^\infty dx \sin(ax^n)\frac{d}{dx}\log\left\{x^n.1+x^{2n}.1+\frac{x^{2n}}{2^{2n}}\ldots\right\}$$

$$= 2\int_0^\infty \Sigma_1^{n-1}\frac{\alpha\sinh 2\pi\alpha x+\beta\sin 2\pi\beta x}{\cosh 2\pi\alpha x-\cos 2\pi\beta x}\sin(ax^n)\,dx, \text{ if } n \text{ be even};$$

and $$= \int_0^\infty \left\{\coth\pi x+2\Sigma_1^{n-2}\frac{\alpha\sinh 2\pi\alpha x+\beta\sin 2\pi\beta x}{\cosh 2\pi\alpha x-\cos 2\pi\beta x}\right\}\sin(ax^n)\,dx,$$
if n be uneven.

Also $q^{1^n}+q^{3^n}+q^{5^n}+\&c.$

$$= \int_0^\infty \Sigma_1^{n-1}\frac{\alpha\sinh\pi\alpha x+\beta\sin\pi\beta x}{\cosh\pi\alpha x-\cos\pi\beta x}\sin(ax^n)\,dx, \text{ if } n \text{ be even};$$

and $$= \int_0^\infty \left\{\tfrac{1}{2}\tanh\tfrac{1}{2}\pi x+\Sigma_1^{n-2}\frac{\alpha\sinh\pi\alpha x+\beta\sin\pi\beta x}{\cosh\pi\alpha x-\cos\pi\beta x}\right\}\sin(ax^n)\,dx, \text{ if } n$$
be uneven.

VI. We have also incidentally the curious results, obtainable at once from the integral written above,

$$\text{(i.)}\ \tfrac{1}{2}+q^{1^n}+q^{2^n}+q^{3^n}+\&c. = \frac{1}{\pi}\int_{-\infty}^{\infty}\sin az\,dx,$$

where z is given by the equation

$$z.1+z^2.1+\frac{z^2}{2^{2n}}.1+\frac{z^2}{3^{2n}}\ldots = e^z.$$

$$\text{(ii.)}\ q^{1^n}+q^{3^n}+q^{5^n}+\&c. = \frac{1}{\pi}\int_0^\infty \sin az\,dx,$$

where z is given by the equation

$$1+z^2.1+\frac{z^2}{3^{2n}}.1+\frac{z^2}{5^{2n}}\ldots = e^z.$$

The equations in I. and II. are the general formulæ, some cases of which I published in the "Messenger of Mathematics," New Series, t. II. pp. 188—190 (1873); the method there adopted is not so good as that used in this paper, and I did not obtain the values for the general products. In a paper "On the summation by Definite Integrals of geometrical series of the Second and Higher Orders," ["Quarterly Journal of Mathematics," t. XI., pp. 328—343 (1871),] I have employed the integral in V. to sum the series $q+q^4+q^9+\&c.$, and

$q+q^8+q^{27}+$&c.,* remarking (p. 339) that "the results (which I have obtained as far as the fourth order inclusive) become so complicated, and their calculation and reduction is so tedious, that it does not seem worth while to work out any of the integrals for higher orders." The theory is thus completed in V. by the absolute expression of the series

of the n^{th} order $\quad q^{1^n}+q^{2^n}+q^{3^n}+$&c.

as a definite integral, the process here employed being far preferable to the direct summation of the series

$$\frac{1}{1^{2n}+x^{2n}}+\frac{1}{2^{2n}+x^{2n}}+\text{&c.}$$

which was the method pursued in the paper in the "Quarterly Journal." Kummer ("Crelle," t. XVII., p. 220) and Schlömilch ("Analytische Studien," Pt. I., p. 158) have both summed the geometrical series of the second order, but their methods do not seem applicable when the exponents are higher powers than squares.

I may add, in conclusion, that it really seems of interest to exhibit as integrals the sums of those series which admit of being so expressed, as, ignoring varieties due to the coefficients, and considering only the exponents, the number of distinct series which can be summed in this manner is very restricted.

December 9th, 1875.

Prof. H. J. S. SMITH, F.R.S., President, in the Chair.

The Auditor (Mr. Perigal) reported that he had audited the Accounts of the Society, comparing them with the vouchers, and had found them correct. It was ordered that the Accounts should be printed in the Proceedings.

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* The value of $q+q^8+q^{27}+$&c. [equation (29)] is wrongly given, that of $q-q^8+q^{27}-$&c. having been written twice over by mistake ("Quarterly Journal," t. XI., p. 339). The correct value, viz.,

$$\tfrac{1}{2}+q+q^8+q^{27}+\text{&c.}=\int_0^\infty\left\{\coth\pi x+\frac{\sinh\pi x+\sqrt{3}\sin(\pi x\sqrt{3})}{\cosh\pi x-\cos(\pi x\sqrt{3})}\right\}\sin(ax^3)\,dx,$$

is derivable at once from the formula at the bottom of p. 338 or from V. in this paper. Also, in equation (30), on p. 339, the fraction whose denominator is $\cosh(\pi x^{\frac{3}{2}})-\cos(\pi x^{\frac{3}{2}}\sqrt{3})$ should be multiplied by 2; and, same page, tenth line from bottom, the exponents 1^r, 2^r &c. are misprinted 1^{rr}, 2^{rr}, &c.

Financial Report for Session 1874–5.

Cr.

	£.	s.	d.
62 Subscriptions, 1874–5	65	2	0
15 „ 1873–4	15	15	0
3 „ 1872–3	3	3	0
2 „ 1871–2	2	2	0
3 „ 1875–6	3	3	0
5 Entrance Fees	5	5	0
Jan. Dividend on Lord Rayleigh's Fund	21	11	4
July Dividend on same	21	11	4
Jan. Dividend on Composition Fund	4	3	8
July Dividend on same	5	3	5
Sale of Proceedings and extra Copies	39	2	9
	£186	2	6

Dr.

	£.	s.	d.
Balance due to Treasurer, brought forward	1	15	6
Postage and Stationery	10	16	10
Journals	5	13	10½
Rent	10	0	0
Attendance	3	7	0
Coal and Gas	1	10	0
Banker's discount on two Irish Cheques	0	1	0
Ditto on Scotch Cheque	0	0	8
Proceedings, Printing &c., paid Messrs. Hodgson & Son—			
Balance due from Session 1873–4 £20 0 0			
Account rendered 13 April, 1875 ... 29 12 6			
	49	12	6
Account rendered Nov. 1875 ...	87	14	0
	170	11	4½
Balance in hand	15	11	1½
	£186	2	6

Audited and found correct, 30 Nov. 1875.

(Signed) H. PERIGAL.

ASSETS AND LIABILITIES.

Cr.

	£.	s.	d.
Balance in hand	15	11	1½
36 Subscriptions, 1874–5	37	16	0
9 „ 1873–4	9	9	0
4 „ 1872–3	4	4	0
	£67	0	1½

6 Life Compositions made in Session 1874–5.

33 „ „ in all, namely—

28 at £10 ..... £280 0 0

5 at £10. 10*s*. 52 10 0

£332 10 0

Invested in £358. 14*s*. 11*d*. Consols.

Dr.

	£.	s.	d.
Printing in progress, about	20	0	0
Postage and Stationery	3	10	10
Balance.....	43	19	3½
	£67	0	1½

Major J. R. Campbell and Prof. G. M. Minchin were elected members of the Society.

Prof. Clifford read a paper on "The Transformation of Elliptic Functions." Prof. Cayley gave some results of his paper "On a System of Algebraical Equations." The President spoke (1) on a Problem of Eisenstein's; (2) on the joint Invariants of two Conics or Quadrics; and (3) on the Equation $P \times D = \text{constant}$, of a Geodesic Line on an Ellipsoid. Mr. Tucker communicated a paper by Prof. H. W. Lloyd Tanner, "On the Solution of certain Partial Differential Equations of the Second Order, having more than two Independent Variables."

The following presents were received:—

"Monatsbericht," Juli and Aug. 1875.

"Bulletin de la Société Mathématique de France," tome iii., Oct., No. 6, Paris.

"Illustrations of the Centimètre-Gramme-Second System of Units:" by Prof. Everett (Physical Society).

"Bulletin des Sciences Mathématiques et Astronomiques," Juillet, Août, 1875.

"Reprint of Mathematics from the 'Educational Times,'" vol. xxiii.

"Crelle," 81 Band, erstes Heft.

Kronecker "über die congruenten Transformationen der bilinearen Formen," Berlin, 1874.

Kronecker "über Schaaren von quadratischen und bilinearen Formen," Berlin, 1874.

Kronecker "über die algebraischen Gleichungen von denen die Theilung der elliptischen Functionen abhängt." (Auszug aus dem "Monatsbericht," 19 Juli 1875.)

Kronecker "über quadratische Formen von negativen Determinante."

Kronecker, "Bemerkungen zur Geschichte des Reciprocitäts-Gesetzes. (Auszug aus dem "Monatsbericht," 19 April 1875.)

"Nautical Almanac," for 1879.

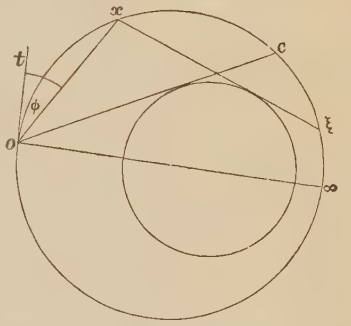
"The Repulsion of Solid Bodies referable to Radiation, a speculation concerning Molecular Physics," by Rev. A. J. Stevens, M.A., Nov. 1875: from the Author.

"On a General Method of producing exact Rectilinear Motion by Link-work," by A. B. Kempe, B.A. (Proceedings of Royal Society, No. 163, 1875): from the Author.

On the Transformation of Elliptic Functions. *By* Prof. Clifford.

[*Read Dec. 9th*, 1875.]

The following communication is an attempt to apply Jacobi's geometrical representation of the addition-theorem in elliptic functions to the theory of their transformation. For this purpose I use the said representation in the following form.



Consider two circles, one of which is wholly within the other, but which are not concentric; as in the figure. The points of the outer circle may be uniquely represented by a parameter x, such that if 0 and ∞ are the points represented by these values respectively, and $0t$ is the tangent at the former, x is proportional to the ratio of the sines of the angles which $0x$ makes with the lines $0t$ and 0∞. Let the angle $x0t = \phi$; then, if we make $x = i \tan\phi$ $(i = \sqrt{-1})$, the values 1, -1 of the parameter will belong to the circular points at infinity. Let then k^{-1}, $-k^{-1}$ be the values belonging to the imaginary points of intersection of the two circles. Through the points 0, x let tangents be drawn to the inner circle, meeting the outer circle in c, ξ; these being so chosen that, when x moves continuously to 0, ξ will move continuously to c. Then Jacobi's theorem is that, if

$$x = \mathrm{sn}\,(u,k), \quad \xi = \mathrm{sn}\,(r,k), \quad c = \mathrm{sn}\,(\gamma,k), \quad \text{then } r = u+\gamma.$$

The extension to any two conics, made by Prof. Cayley, may be put into the same form. The representation of each point of a conic by a parameter is determinate when we know the parameters of any three points. Now the four intersections of two conics U, V may be divided into pairs in three ways, and will so determine three involutions upon the conic U. Let one of these be chosen, and let the parameters 0 and ∞ be assigned to its united points. Then, if the value 1 be assigned to one of the intersections, -1 will belong to another of them; and the remaining two will have parameters equal in magnitude but of contrary signs. Call these $\pm k^{-1}$, and draw the tangents as before; then Jacobi's theorem remains true, except that we must write $\pm\mathrm{sn}^{-1}\xi = \mathrm{sn}^{-1}x + \mathrm{sn}^{-1}c$; the sign of $\mathrm{sn}^{-1}\xi$ depending on the reality of the common tangents.

The proof of it depends on the symmetrical (2, 2) correspondence,

considered by Euler, in relation to the addition of elliptic integrals. Given that the relation between the points x and ξ is that the line $x\xi$ touches the conic V, it is clear that to one position of x correspond two positions of ξ; and to one position of ξ, two positions of x; or, the points have a (2, 2) correspondence. Hence the equation connecting the quantities x, ξ must be of the second order in each; and it must obviously be symmetrical. Let the equation be

$$(ax^2+2bx+c)\,\xi^2+2\,(bx^2+2b'x+c')\,\xi+(cx^2+2c'x+c'')=0\,;$$

or, which is the same thing,

$$(a\xi^2+2b\xi+c)\,x^2+2\,(b\xi^2+2b'\xi+c')\,x+(c\xi^2+2c'\xi+c'')=0.$$

The values of x which make the two corresponding values of ξ coincide are given by the equation

$$\mathrm{X}=(ax^2+2bx+c)\,(cx^2+2c'x+c'')-(bx^2+2b'x+c')^2=0,$$

and similarly the values of ξ which make the two corresponding values of x coincide are given by $\Xi=0$, where Ξ is the same function of ξ that X is of x. Now, by differentiating the original equation, we easily find $dx:\sqrt{\mathrm{X}}=d\xi:\sqrt{\Xi}$.

The roots of the equation $\mathrm{X}=0$ are clearly the parameters of the points of intersection of the two conics; for these are the only points on U from which two coincident tangents can be drawn to V. If, then, the parameters of these points have been made equal to ±1, $\pm k^{-1}$, X must be proportional to $(1-x^2)\,(1-k^2x^2)$, and the differential equation

becomes $$\frac{dx}{\sqrt{(1-x^2)\,(1-k^2x^2)}}=\frac{d\xi}{\sqrt{(1-\xi^2)\,(1-k^2\xi^2)}}\,;$$

whence, since, when $x=0$, $\xi=c$,

$$\pm\,\mathrm{sn}^{-1}x+\mathrm{sn}^{-1}c=\mathrm{sn}^{-1}\xi,$$

which is the theorem in question.

If we change V into $\mathrm{U}+\sigma\mathrm{V}$, and allow σ to vary, this varying conic will always have the same intersections with U, and therefore k will be constant; but c will depend upon the value of σ. It is clear that c^2 is given uniquely when σ is given, but, when c is given, there are two values of σ. When $c=0$, the two conics must touch at 0, and therefore must coincide; thus both values of σ vanish. When $c=\infty$, the conic $\mathrm{U}+\sigma\mathrm{V}$ becomes a pair of lines intersecting on the line $0\,\infty$; let α and β be the values of σ which belong to these, and which are, of course, roots of the equation $\square\,(\mathrm{U}+\sigma\mathrm{V})=0$.

Then we must have $c^2=\dfrac{m\sigma^2}{(\sigma-\alpha)\,(\sigma-\beta)}$,

where m is an undetermined constant.

Suppose, now, that a polygon is inscribed in U by the following process; a tangent is drawn from x to a conic σ_1, meeting U again in x_1; then from x_1 to a conic σ_2, meeting U in x_2, and so on; finally, let x_{n-1} be joined to x. If $c_1, c_2 \ldots\ldots c_{n-1}$ be the constants belonging to these conics respectively, we shall have

$$\begin{aligned} \pm \mathrm{sn}^{-1} x_1 &= \mathrm{sn}^{-1} x + \mathrm{sn}^{-1} c_1, \\ \pm \mathrm{sn}^{-1} x_2 &= \mathrm{sn}^{-1} x_1 + \mathrm{sn}^{-1} c_2; \\ \&c. &\quad\quad \&c. \\ \pm \mathrm{sn}^{-1} x_{n-1} &= \mathrm{sn}^{-1} x_{n-2} + \mathrm{sn}^{-1} c_{n-1}; \end{aligned}$$

whence, by addition, with proper changes of sign,

$$\begin{aligned} \pm \mathrm{sn}^{-1} x_{n-1} &= \mathrm{sn}^{-1} x + \mathrm{sn}^{-1} c_1 \pm \mathrm{sn}^{-1} c_2 \pm \ldots \pm \mathrm{sn}^{-1} c_{n-1} \\ &= \mathrm{sn}^{-1} x + \mathrm{sn}^{-1} c_n, \text{ suppose.} \end{aligned}$$

From this equation it appears that the last side of the polygon will always touch the same conic of the series $\mathrm{U} + \sigma \mathrm{V}$, wherever the starting point x is taken. Or, if a polygon be inscribed in U, and move so that all but one of its sides touch conics of the series $\mathrm{U} + \sigma \mathrm{V}$, then the remaining side will also touch a conic of the series. This theorem of Poncelet's is proved in this way by Jacobi. It is to be noticed that the signs of the quantities c depend upon which tangent is drawn from the corresponding point x; the two tangents belong in a definite way to the two tangents from 0. The final value of c_n being thus determined, one of the two conics belonging to it is singled out by the sign given to $\mathrm{sn}^{-1} x_{n-1}$. In fact, the whole series of conics $\mathrm{U} + \sigma \mathrm{V}$ is divided into three parts by the three line-pairs it contains. For two conics in the same part of the series the signs of $\mathrm{sn}^{-1} \xi$ are certainly the same; for conics in different parts they may be different.

And it appears that the conics mentioned in the theorem may belong to any part of the series if the signs be properly chosen in each of the equations. It is remarked by Jacobi that with these restrictions the position of x_{n-1} does not depend upon the order in which the conics are taken.

Now suppose two such polygons to be drawn with the same system of conics, x being continuously moved to ξ, and at the same time x_1 to ξ_1, &c. We shall then have the equations

$$\begin{aligned} \pm \mathrm{sn}^{-1} x_1 &= \mathrm{sn}^{-1} x + \mathrm{sn}^{-1} c_1, \\ \pm \mathrm{sn}^{-1} \xi_1 &= \mathrm{sn}^{-1} \xi + \mathrm{sn}^{-1} c_1, \end{aligned}$$

the signs being the same in both. Consequently

$$\pm \left(\mathrm{sn}^{-1} x_1 - \mathrm{sn}^{-1} \xi_1 \right) = \mathrm{sn}^{-1} x - \mathrm{sn}^{-1} \xi,$$

or the lines $x_1 \xi_1$, $x\xi$ touch the same conic of the series $\mathrm{U} + \sigma \mathrm{V}$.

Proceeding in this way, we may show that

$$\pm(\mathrm{sn}^{-1}x_r - \mathrm{sn}^{-1}\xi_r) = \mathrm{sn}^{-1}x - \mathrm{sn}^{-1}\xi;$$

whence it appears that the lines joining corresponding vertices of the two polygons all touch the same conic of the series when the n conics touched by the sides belong to the same part; but if they belong to different parts, each joining line touches one of two conics which harmonically divide the pairs of lines $\mathrm{U}+\alpha\mathrm{V}$, $\mathrm{U}+\beta\mathrm{V}$.

Attending now only to the first case, it will be convenient to re-state the two theorems together, as follows:—

If a polygon be inscribed in a conic U *so that all its sides but one touch conics of the series* $\mathrm{U}+\sigma\mathrm{V}$, *the remaining side will also touch a conic of the series.*—(Poncelet's Theorem.)

When all these conics can pass continuously into one another without breaking up into two straight lines, the lines joining corresponding vertices of two such polygons will all touch a conic of the series.

Let us now consider the particular case in which all the sides of the moving polygon touch the same conic. Here the second theorem is true without restriction; the lines joining corresponding vertices of two such polygons will always touch one conic passing through the intersections of the other two. In this case also *the vertices of the variable polygon determine upon the conic* U *an involution of the* n^{th} *order;* that is to say, if the parameters of the vertices of one polygon be determined by an equation $p_n = 0$ of the n^{th} order, and those of another polygon by an equation $q_n = 0$, then the vertices of any third polygon will be determined by an equation $p_n - yq_n = 0$, where y is a variable quantity, which we may call the parameter of the polygon. The relation between y, the parameter of the polygon, and x the parameter of any one of its vertices is $y = p_n : q_n$, where p_n, q_n are rational integral functions of the n^{th} order in x.

Suppose, then, the relation between U and V to be such that a polygon of n sides may be inscribed in U and circumscribed to V. Let $x, x_1 \ldots x_{n-1}$ be the vertices of such a polygon; then, if $x = \mathrm{sn}\,u$, we must have $x_1 = \mathrm{sn}\,(u+\gamma)$, $x_2 = \mathrm{sn}\,(u+2\gamma)$, $x_3 = \mathrm{sn}\,(u+3\gamma)$ ..... $x_{n-1} = \mathrm{sn}\,[u+(n-1)\,\gamma]$, and consequently $x = \mathrm{sn}\,(u+n\gamma)$. Therefore $n\gamma$ is a period of the elliptic function; and the number of conics of the series $\mathrm{U}-\sigma\mathrm{V}$, which can be inscribed in n-gons inscribed in U is equal to the number of periods whose n^{th} parts are not congruent, that is, for n a prime number it is $n+1$.

Now let another polygon be drawn having a vertex at ξ, and let η be its parameter. Then the lines $x\xi$, $x_1\xi_1$ ... &c. will all touch a conic W. Let this conic be held fixed, and the two polygons moved so that the lines joining corresponding vertices always touch W. Then to every value of y will belong two values of η, and *vice versâ*, and this

(2, 2) correspondence is symmetrical. Hence *a symmetrical* (2, 2) *correspondence between individual vertices implies a symmetrical* (2, 2) *correspondence between the polygons.*

Now, the parameter of every polygon is determined when we know the parameters of three polygons. Let the parameters of the polygons which have a vertex at 0, 1, and ∞ be made equal to 0, 1, and ∞ respectively; this amounts to saying that $p_n = 0$ has a root 0, $q_n = 0$ has a root ∞, and $p_n - q_n = 0$ has a root 1. It is clear, then, from the symmetry of the figure, that y must be an odd function of x, so that $p_n + q_n = 0$ will have a root -1. This amounts to saying that p_n is x multiplied by a rational integral function of x^2, and q_n (which is really only of the order $n-1$) is another rational integral function of x^2. This being so, let $\pm\lambda^{-1}$ be the parameter of those polygons which have vertices at the remaining two points of intersection of the conics. Then the quantities y and η are connected by a symmetrical (2, 2) correspondence such that the values of y which give equal values for η are ± 1, $\pm\lambda^{-1}$. Therefore, if $y = \mathrm{sn}\,(u', \lambda)$, we must have $\eta = \mathrm{sn}\,(u'+\delta, \lambda)$, where δ is a constant.

We have arranged that y is divisible by x or $\mathrm{sn}\,u$, by making 0 the parameter of that polygon which has a vertex at the point 0. It appears thus that y must also be divisible by $x_1, x_2 \ldots \ldots x_{n-1}$, since, when any one of these is zero, y vanishes. We may write, therefore, $y = mxx_1 \ldots x_{n-1}$, where m is a constant, since y is only infinite when one or other of the x is infinite. The products $x_1 x_{n-1}$, $x_2 x_{n-2}$, &c., are given rationally as ratios of quadratic functions of x by the original equation of (2, 2) correspondence. To determine m, we have

$$y = m\,\mathrm{sn}\,u\,\mathrm{sn}\,(u+\gamma)\,\mathrm{sn}\,(u+2\gamma)\,\ldots\,\mathrm{sn}\,\{u+(n-1)\,\gamma\}\quad[n\gamma = a\mathrm{K}+bi\mathrm{K}'];$$

but, since $y = 1$, when $x = \mathrm{sn}\,u = 1$, or $u = \mathrm{K}$, we have also

$$1 = m\,\mathrm{sn}\,\mathrm{K}\,\mathrm{sn}\,(\mathrm{K}+\gamma)\,\mathrm{sn}\,(\mathrm{K}+2\gamma)\,\ldots\ldots\,\mathrm{sn}\,[\mathrm{K}+(n-1)\,\gamma],$$

and therefore $$y = \frac{\mathrm{sn}\,u\,\mathrm{sn}\,(u+\gamma)\,\ldots\ldots\,\mathrm{sn}\,[u+(n-1)\,\gamma]}{\mathrm{sn}\,\mathrm{K}\,\mathrm{sn}\,(\mathrm{K}+\gamma)\ldots\ldots\mathrm{sn}\,[\mathrm{K}+(n-1)\,\gamma]};$$

the denominator may, of course, be simplified, as in Jacobi's formula, but for present purposes it may be left as it stands. Now λ^{-1} is the value of y when $x = k^{-1}$, or when $u = \mathrm{K}+i\mathrm{K}'$,

$$\frac{\lambda}{k} = \frac{\mathrm{sn}\,\mathrm{K}\,\mathrm{sn}\,(\mathrm{K}+\gamma)\,\ldots\,\mathrm{sn}\,[\mathrm{K}+(n-1)\,\gamma]}{\mathrm{sn}\,(\mathrm{K}+i\mathrm{K}')\,\mathrm{sn}\,(\mathrm{K}+i\mathrm{K}'+\gamma)\ldots\mathrm{sn}\,[\mathrm{K}+i\mathrm{K}'+(n-1)\gamma]},$$

which, again, may be easily reduced.

If we now assume that $u = \mathrm{M}u'$, where $y = \mathrm{sn}\,(u', \lambda)$ and M is a constant, we may determine M by observing that it is the value of $x : y$

when both of them are zero. Namely we have

$$M = \frac{\mathrm{sn}\,(K+\gamma)\,\mathrm{sn}\,(K+2\gamma)\ldots\mathrm{sn}\,[K+(n-1)\,\gamma]}{\mathrm{sn}\,\gamma\,\mathrm{sn}\,2\gamma\ldots\mathrm{sn}\,(n-1)\,\gamma}.$$

And thus, with the exception of the assumption just made, the theory of transformation is established by means of that of the in-and-circumscribed polygon.

Let us now endeavour to generalise the theory of the in-and-circumscribed polygon. In the first place, we may observe that every involution of the third order gives rise to an elliptic transformation; for two triangles inscribed in a conic are always circumscribed to the same conic.* For convenience, we will now consider the involution as determined on the conic V; namely, the triangles form groups of three tangents which are in involution. And we may now generalize our theorem as follows:—

If a complete n*-gram move with its sides touching a conic so that they form groups of* n *tangents in involution, the locus of the* $\frac{1}{2}$ n (n−1) *vertices is a curve of order* n−1.

For, consider the number of points which the curve has in common with any one tangent of the conic. It determines uniquely in the involution the group of n tangents to which it belongs, and can have no other point on the locus of intersections except those $n-1$ in which it meets the other $n-1$ tangents of this group. Therefore &c.

If, in a curve of order n−1, *it is possible to inscribe one complete* n*-gram whose sides all touch a conic, then it is possible to inscribe a singly infinite number, and the sides determine upon the conic an involution of the* nth *order.*

Let A, B, C, ... N = 0 be the equations of the sides of the complete n-gram, then the equation of the curve may be written

$$\frac{\alpha}{A}+\frac{\beta}{B}+\frac{\gamma}{C}+\ldots\ldots+\frac{\nu}{N}=0\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

where α, β, γ . ν are constants. But since A is a tangent to the conic, its equation may be written in the form $X+aY+a^2Z=0$, where a is the parameter of the tangent and X, Y, Z three fixed lines. The condition for three tangents to meet in a point is

$$0=\begin{vmatrix}1, & x, & x^2\\ 1, & y, & y^2\\ 1, & z, & z^2\end{vmatrix}=(y-z)\,(z-x)\,(x-y)\ \ldots\ldots\ldots\ldots\ldots(2),$$

* Every substitution $y=\frac{U}{V}$, where U, V are cubic functions of x, has an elliptic differential which it transforms. The cubic forms U, V are first polars of two points in regard to a single quartic form F (Gundelfinger, "Math. Annalen"). Let X = 0 give the four points whose first polars in regard to F have a square factor; then $dx : \sqrt{X}$ is the elliptic differential required. We have $X = jF + iH$, where H is the Hessian of F, and i, j the quadrinvariant and cubinvariant.

where x, y, z are their parameters. Hence the condition that the tangents whose parameters are x, y shall meet on the curve (1) is

$$\frac{\alpha}{(a-x)(a-y)} + \frac{\beta}{(b-x)(b-y)} + \ldots\ldots + \frac{\nu}{(u-x)(u-y)} = 0 \quad \ldots(3);$$

or, which is the same thing,

$$\Sigma \frac{\alpha}{a-x} = \Sigma \frac{\alpha}{a-y},$$

and therefore, if tangents xy and also xz meet on the curve, it follows that yz will meet on the curve. Starting, then, from any arbitrary tangent x, we can find the $n-1$ points in which this meets the curve, and draw from them $n-1$ other tangents; the intersection of any two of these will lie on the curve, by what we have just proved. That is to say, we can inscribe a complete n-gram which shall have for one side any arbitrary tangent of the conic.

Now, suppose that x is given, then, regarding the equation (3) as determining $n-1$ values of y, we can find the product of the roots. Namely, it is

$$= \Sigma \frac{\alpha \,.\, \Pi a}{a\,(a-x)} : \Sigma \frac{\alpha}{a-x}, \quad \Pi a = a \,.\, b \,.\, c \ldots\ldots n.$$

If we multiply this product by x, we obtain the product of the parameters of all the tangents forming a complete n-gram; let this be called λ; then, observing that

$$\frac{ax}{a\,(a-x)} = \frac{\alpha}{a} - \frac{\alpha}{a-x},$$

we shall find

$$\left(1 - \frac{\lambda}{\Pi a}\right) \Sigma \frac{\alpha}{a-x} = \Sigma \frac{\alpha}{a} \quad \ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Now this is an equation of the n^{th} order in x, the roots of which are the parameters of the sides of a complete n-gram; and λ is the product of these roots. Since λ is linearly involved, the equation shows that these groups of n tangents form an involution of the n^{th} order, and that λ is proportional to the parameter of such a group in the involution when the groups containing the tangents 0, ∞ are made to have the parameters 0, ∞ respectively. It appears also that the sum of the roots, sum of their products in pairs, &c., are each given as linear functions of λ, and might each be used as parameters of the involution.

We shall now endeavour to find an expression for ABC ... N.

Let $1 - \lambda\,(\Pi a)^{-1} : \Sigma\, \alpha\, a^{-1}$ be called θ, and let $\Pi_a\,(x-a)$ mean the product $(x-a)\,(x-b)\,\ldots\,(x-n)$, then the equation for x may be put into the form $\Pi_a\,(x-a) + \theta\,\Pi_a\,(x-a) \,.\, \Sigma a\,(x-a)^{-1} = 0 \ldots\ldots\ldots\ldots(5)$, where the first term is of order n, and the second of order $n-1$ in x. By a slight change of notation, let the n roots be called $x_1, x_2, \ldots x_n$, and let

$\Pi_x(y-x)$ mean $(y-x_1)(y-x_2)\ldots(y-x_n)$. Then we have

$$(x-x_1)(x-x_2)\ldots\ldots(x-x_n) = \Pi_a(x-a)+\theta\Pi_a(x-a)\,\Sigma\alpha\,(x-\alpha)^{-1};$$

and therefore

$$\Pi_x(y-x) = \Pi_a(y-a)+\theta\Pi_a(y-a)\,\Sigma\alpha\,(y-\alpha)^{-1}.$$

Multiplying together two such equations, we obtain

$$\begin{aligned}\Pi_x(y-x)\,.\,\Pi_x(z-x) = {}& \\ & \Pi_a(y-a)\,.\,\Pi_a(z-a) \\ & +\theta\,\Pi_a(y-a)\,.\,\Pi_a(z-a)\,.\,[\Sigma\alpha\,(y-\alpha)^{-1}+\Sigma\alpha\,(z-\alpha)^{-1}] \\ & +\theta^2\Pi_a(y-a)\,.\,\Pi_a(z-a)\,.\,\Sigma\alpha\,(y-\alpha)^{-1}\,.\,\Sigma\alpha\,(z-\alpha)^{-1}.\end{aligned}$$

Now, if we examine this equation, we shall find that the left-hand member is A′B′C′ ... N′, where A′ = 0, B′ = 0 ... are the tangents which make up the n-gram belonging to the parameter θ; the first term on the right is ABC ... N, the n-gram for $\theta = 0$, and the last term is $\theta^2 Z_1 Z_2 \ldots Z_n$, the n-gram for $\theta = \infty$, which has the line Z = 0 in it. Thus we may write the equation

$$\Pi A' = \Pi A+\theta P_n+\theta^2\Pi Z \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6),$$

and it only remains to find the nature of the curve P_n of the n^{th} order. We may see from its equation,

$$0 = \Pi_a(y-a)\,\Pi_a(z-a)\,[\Sigma\alpha\,(y-\alpha)^{-1}+\Sigma\alpha\,(z-\alpha)^{-1}],$$

that it passes through the points of contact and all the intersections of the n tangents ΠA; and then it is clear, from the symmetry, that it must pass through the points of contact and all the intersections of the tangents ΠZ. But perhaps the simplest way is to consider the envelope of the n-gram $\Pi A'$, which we know must consist of the conic K_2 once, and the locus of the nodes C_{n-1} twice; thus we shall have

$$4\,\Pi A\,.\,\Pi Z-P_n^2 = K_2\,C_{n-1}^2$$

to a factor *près*, and this equation gives at once the intersections of P_n with K_2, and with the n-gram.

We may now state the following propositions:—

Given any two in-and-circumscribed n*-grams* ΠX *and* ΠZ, *there exists always a curve* P_n *of order* n *which passes through their* n(n−1) *vertices and their* 2n *points of contact with the conic.*

The equation of any other n*-gram may be written in the form*

$$0 = \Pi X+\lambda P_n+\lambda^2\Pi Z.$$

The relation between λ, *the parameter of the* n*-gram, and* x *the parameter of one of its sides, is*

$$(\Pi a-\lambda)\,\Sigma\alpha\,(a-x)^{-1} = \Pi a\,.\,\Sigma\alpha a^{-1},$$

and λ *is the product of the roots of this equation.*

I have here taken ΠX, $(X_1 X_2 \ldots X_n)$ for the first n-gram, corres-

ponding to $\lambda = 0$, instead of ΠA, corresponding to $\lambda = \Pi\alpha$ or $\theta = 0$.

We may show, conversely, that if the envelope of

$$0 = P + \lambda Q + \lambda^2 R,$$

where $P = 0$, $Q = 0$, $R = 0$ are three curves of the n^{th} order, is $4PR - Q^2 = K_2 C^2$, C being of order $n-1$; then P and R are each an assemblage of n straight lines. For the curve P has nodes on all its intersections with C, since $4PR = Q^2 + K_2 C^2$; that is, $\frac{1}{2}n\,(n-1)$ nodes, so that it must consist of n straight lines.

[This point of view immediately suggests the extension of the whole theory to quadric surfaces. If the envelope of $P + \theta Q + \phi R + \theta\phi S$ is $PS - QR = K_2 C^2$, where P, Q, R, S are of order n, and C of order $n-1$, each of the surfaces P, Q, R, S will meet the quadric K_2 in two curves of the n^{th} order, and therefore will have $\frac{1}{2}n^2$ contacts with it; and similarly will meet C_{n-1} in two curves of order $\frac{1}{2}n\,(n-1)$, which intersect in $\frac{1}{4}n^2\,(n-1)$ points; these are not contacts, but nodes on the surface P. We thus get a theory of surfaces of order n, having $\frac{1}{2}n^2$ contacts with a quadric surface, and $\frac{1}{4}n^2\,(n-1)$ nodes on a fixed surface of order $n-1$. It appears that n must be even, and of course the variable surface is subject to other conditions. Thus, in the case of a cone doubly tangent to the quadric, and having its vertex on a fixed plane, it has also to pass through two fixed points on the plane and four on the quadric.

The application of the identity $4PR - Q^2 = K_2 C^2$ to surfaces only reproduces the theory of the plane conic.]

At any point of intersection of the two n-grams,

$$0 = \Pi X + x P_n + x^2 \Pi Z,$$

$$0 = \Pi X + y P_n + y^2 \Pi Z,$$

we shall have $\quad xy : -x-y : 1 = \Pi X : P_n : \Pi Z.$

Consequently, any symmetrical (m, m) correspondence between the two n-grams expresses that they intersect on a curve of order mn, namely $(\Pi X, -P_n, \Pi Z)^m$, if the equation of the (m, m) correspondence is $(xy, x+y, 1)^m$.

If we substitute the values $xy : -x-y : 1$ for $\Pi X : P_n : \Pi Z$, in the equation of a third n-gram $0 = \Pi X + z P_n + z^2 \Pi Z$, we shall get simply $(z-x)\,(z-y) = 0$. Consider, then, m different n-grams ΠA_1, $\Pi A_2 \ldots \Pi A_m$, and form the curve of order $n\,(m-1)$

$$\frac{\beta_1}{\Pi A_1} + \frac{\beta_2}{\Pi A_2} + \ldots = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (7);$$

if the n-grams x, y meet on this curve we shall have

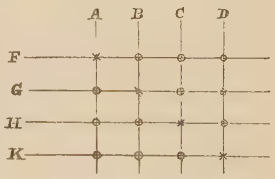
$$\Sigma \frac{\beta}{(\alpha - x)\,(\alpha - y)} = 0;$$

or, what is the same thing,

$$\Sigma \frac{\beta}{\alpha - x} = \Sigma \frac{\beta}{\alpha - y},$$

where $a_1\ a_2 \ldots a_m$ are the parameters of the n-grams. It follows, as before, that a singly infinite number of groups of n-grams can be totally inscribed in the curve (7); a group being totally inscribed when all the intersections of any two n-grams of the group are on the curve.

So far we have dealt only with totally inscribed n-grams; and, as this case is represented only by the triangle when the curve of inscription is a conic, it might seem that there should be more general theorems corresponding to the inscription of polygons of a greater number of sides in a conic. But, in fact, the case of total inscription is the general case, and all others are cases of decomposition of the curve C_{n-1}. Consider, for example, a hexagon inscribed in a conic. If we produce all the sides, we shall get nine more intersections; three of these lie on a straight line, and the other six on a conic which circumscribes two triangles. These two conics and the straight line make up the curve C_5 of the fifth order. Again, let eight lines A, B, C, D F, G, H, K touch a conic, and let the twelve points marked (o) in the figure touch a cubic; then the octagram may be moved round the conic so as to keep these twelve points on the cubic. The points marked (×) will lie on a fixed straight line, and a second cubic will pass through the intersections of A, B, C, D among themselves, and of F, G, H, K among themselves. These two cubics and the straight line make up the curve C_7 of the seventh order; and it is easy to see that there is an analogous case for any even number of lines. In order that a porismatic polygram may be inscribed in a curve, it is necessary that either the order of the curve or the number of sides should be even.



On a System of Equations connected with Malfatti's Problem.

By Prof. Cayley.

[*Read December 9th*, 1875.]

I consider the equations

$$\begin{aligned} X, &= by^2+cz^2-2fyz-a(bc-f^2), &= 0,\\ Y, &= cz^2+ax^2-2gzx-b(ca-g^2), &= 0,\\ Z, &= ax^2+by^2-2hxy-c(ab-h^2), &= 0, \end{aligned}$$

where the constants (a, b, c, f, g, h) are such that

$$K, = abc-af^2-bg^2-ch^2+2fgh, = 0.$$

Hence, writing as usual (A, B, C, F, G, H) to denote the inverse coefficients $(bc-f^2, ca-g^2, ab-h^2, gh-af, hf-bg, fg-ch)$,
we have $(A, B, C, F, G, H \mathcal{X} x, y, z)^2 =$ the square of a linear function, $= (\alpha x+\beta y+\gamma z)^2$ suppose; that is,

$$(A, B, C, F, G, H) = (\alpha^2, \beta^2, \gamma^2, \beta\gamma, \gamma\alpha, \alpha\beta).$$

It is to be shown that the three quadric surfaces $X=0$, $Y=0$, $Z=0$ intersect in a conic Θ lying in the plane $\alpha\alpha x+b\beta y+c\gamma z=0$, and in two points I, J; or more completely, that

surfaces Y, Z meet in conic Θ and a conic P,
„ Z, X „ „ „ Q,
„ X, Y „ „ „ R,

where the conics P, Q, R each pass through the two points I, J, and meet the conic Θ in two points, viz.,

conics P, Θ meet in two points P_1, P_2,
„ Q, Θ „ „ Q_1, Q_2,
„ R, Θ „ „ R_1, R_2.

For this purpose, writing

$$\nabla = Aa-Ff, \ =Bb-Gg, \ =Cc-Hh, \ =abc-fgh,$$

$$\Omega = \tfrac{1}{2}(X+Y+Z), = ax^2+by^2+cz^2-fyz-gzx-hxy-\nabla,$$

$$\theta = a\alpha x+b\beta y+c\gamma z,$$

$$\xi = \frac{Aa}{\alpha}x+\frac{Ff}{\beta}y+\frac{Ff}{\gamma}z,$$

$$\eta = \frac{Gg}{\alpha}x+\frac{Bb}{\beta}y+\frac{Gg}{\gamma}z,$$

$$\zeta = \frac{Hh}{\alpha}x+\frac{Hh}{\beta}y+\frac{Cc}{\gamma}z,$$

then we have identically $\quad a\mathrm{A}\Omega-\nabla X = \theta\xi,$

$$b\mathrm{B}\Omega-\nabla Y = \theta\eta,$$

$$c\mathrm{C}\Omega-\nabla Z = \theta\zeta.$$

In fact, the first of these equations written at full length is

$$a\mathrm{A}\,(ax^2+by^2+cz^2-fyz-gzx-hxy-\nabla)-\nabla\,(by^2+cz^2-2fyz-a\mathrm{A})$$
$$= (a\alpha x+b\beta y+c\gamma z)\left(\frac{Aa}{\alpha}x+\frac{Ff}{\beta}y+\frac{Ff}{\gamma}z\right),$$

where on the left hand side the constant term is $=0$; and comparing first the coefficients of x^2, y^2, z^2, on the two sides respectively, these are Aa^2, $(Aa-\nabla)\,b$, $(Aa-\nabla)\,c$, and Aa^2, Ffb, Ffc, which are equal. Comparing the coefficients of yz, zx, xy, the equations which remain to be

verified are

$$-(a\mathrm{A}-2\nabla)f = \mathrm{F}f\left(c\frac{\gamma}{\beta}+\frac{b\beta}{\gamma}\right),$$

$$-a\mathrm{A}g \qquad = \mathrm{F}af.\frac{\alpha}{\gamma}+\mathrm{A}ac.\frac{\gamma}{\alpha},$$

$$-a\mathrm{A}h \qquad = \mathrm{F}af.\frac{\alpha}{\beta}+\mathrm{A}ab.\frac{\beta}{\alpha};$$

or, as these may be written,

$$(-a\mathrm{A}-2\nabla)\,\beta\gamma = \mathrm{F}\,(c\gamma^2+b\beta^2),$$

$$-\mathrm{A}g\gamma\alpha \qquad = \mathrm{F}f\alpha^2+\mathrm{A}c\gamma^2,$$

$$-\mathrm{A}h\alpha\beta \qquad = \mathrm{F}f\alpha^2+\mathrm{A}b\beta^2;$$

and substituting for α^2, β^2, γ^2, $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$ their values, these may be verified without difficulty.

It thus appears that the equations of the three quadric surfaces may be written in the form

$$\mathrm{A}a\Omega-\theta\xi=0,\ b\mathrm{B}\Omega-\theta\eta=0,\ c\mathrm{C}\Omega-\theta\zeta=0;$$

and we thus obtain the conics Θ, P, Q, R as the intersections of the surface $\Omega=0$ by the four planes

$$\theta=0,\ \frac{\eta}{\mathrm{B}b}-\frac{\zeta}{\mathrm{C}c}=0,\ \frac{\zeta}{\mathrm{C}c}-\frac{\xi}{\mathrm{A}a}=0,\ \frac{\xi}{\mathrm{A}a}-\frac{\eta}{\mathrm{B}b}=0,$$

respectively. There is no difficulty in verifying that the conics intersect as mentioned above, and that the coordinates of their points of intersection are

$$\mathrm{P},\ \mathrm{P}_1 : \left(\sqrt{bc},\ \frac{ch}{\sqrt{bc}},\ \frac{bg}{\sqrt{bc}}\right),\ \left(-\sqrt{bc},\ -\frac{ch}{\sqrt{bc}},\ -\frac{bg}{\sqrt{bc}}\right);$$

$$\mathrm{Q},\ \mathrm{Q}_1 : \left(\frac{ch}{\sqrt{ca}},\ \sqrt{ca},\ \frac{af}{\sqrt{ca}}\right),\ \left(-\frac{ch}{\sqrt{ca}},\ -\sqrt{ca},\ -\frac{af}{\sqrt{ca}}\right);$$

$$\mathrm{R},\ \mathrm{R}_1 : \left(\frac{bg}{\sqrt{ab}},\ \frac{af}{\sqrt{ab}},\ \sqrt{ab}\right),\ \left(-\frac{bg}{\sqrt{ab}},\ -\frac{af}{\sqrt{ab}},\ -\sqrt{ab}\right);$$

$$\mathrm{I},\ \mathrm{J} : (f,\ g,\ h),\qquad (-f,\ -g,\ -h).$$

In a paper "On a system of Equations connected with Malfatti's Equation and on another Algebraical System," Camb. and Dublin Math. Journal, vol. IV. (1849), pp. 270—275, I considered a system of equations which, writing therein $\theta=1$, and changing the signs of (f, g, h), are the equations here considered, $\mathrm{X}=0$, $\mathrm{Y}=0$, $\mathrm{Z}=0$, only the constants (a, b, c, f, g, h) are not connected by the equation $\mathrm{K}=0$, but are perfectly arbitrary. The three quadric surfaces intersect therefore in 8 points, the coordinates of which are obtained in the paper just

referred to, viz., making the above changes of notation, the values are

$$x^2 = \frac{1}{2a}(abc+fgh-f\sqrt{BC}+g\sqrt{CA}+h\sqrt{AB}),$$

$$y^2 = \frac{1}{2b}(abc+fgh+f\sqrt{BC}+g\sqrt{AB}+h\sqrt{AB}),$$

$$z^2 = \frac{1}{2c}(abc+fgh+f\sqrt{BC}+g\sqrt{CA}-h\sqrt{AB}),$$

$$yz = \tfrac{1}{2}(gh+af+\sqrt{BC}),$$

$$zx = \tfrac{1}{2}(hf+bg+\sqrt{CA}),$$

$$xy = \tfrac{1}{2}(fg+ch+\sqrt{AB});$$

where the radicals are such that $\sqrt{BC}\,.\,\sqrt{CA}\,.\,\sqrt{AB} = ABC$, so that the system $(x^2, y^2, z^2, yz, zx, xy)$ has four values only, and consequently (x, y, z) has eight values.

It is very remarkable that, introducing the foregoing relation $K=0$, there is not in the solution any indication that the intersection has become a conic and two points, but the solution gives eight determinate points, viz., the before mentioned points P, P_1, Q, Q_1, R, R_1, and I, J.

To develope the solution, remark that, in virtue of the relation in question, we have

$$\sqrt{BC} = \pm F, \quad \sqrt{CA} = \pm G, \quad \sqrt{AB} = \pm H,$$

where the signs must be such that the product is $= FGH$ (viz., they must be all positive, or else one positive and the other two negative); for taking the product to be $+FGH$, the equations give

$$0 = ABC-FGH,$$

that is, $0 = A(BC-F^2)-F(GH-AF), = K(Aa-Ff), = K\nabla,$

which is true in virtue of the relation $K=0$. Taking the signs all positive, we have for $x^2, y^2, z^2, yz, zx, xy$, the values $f^2, g^2, h^2, gh, hf, fg$, viz., we have thus the points

$$(f, g, h), \quad (-f, -g, -h),$$

which are the points I, J. Taking the signs one positive and the other two negative, say $\sqrt{BC} = F$, $\sqrt{CA} = -G$, $\sqrt{AB} = -H$, we find for $x^2, y^2, z^2, yz, zx, xy$ the values $bc, \frac{ch^2}{b}, \frac{bg^2}{c}, gh, bg, ch$, viz., we have thus the points

$$\left(\sqrt{bc}, \frac{ch}{\sqrt{bc}}, \frac{bg}{\sqrt{bc}}\right), \quad \left(-\sqrt{bc}, -\frac{ch}{\sqrt{bc}}, -\frac{bg}{\sqrt{bc}}\right),$$

which are the points P, P_1; and the other two combinations of sign give of course the points Q, Q_1 and R, R_1 respectively.

If the coefficients (a, b, c, f, g, h), instead of the foregoing relation $K = 0$, satisfy the relation

$$abc-af^2-bg^2-ch^2-2fgh = 0, \quad \text{say } K' = 0,$$

the quadric surfaces intersect in 8 points, the coordinates of which are given by the general formulæ, but the expressions assume a very simple form. Writing for shortness

$$F' = gh + af, \quad G' = hf + bg, \quad H' = fg + ch,$$

then in virtue of the assumed relation

$$\sqrt{BC} = \pm F', \quad \sqrt{CA} = \pm G', \quad \sqrt{AB} = \pm H',$$

where the signs are such that the product of the three terms is positive, viz., they must be all positive, or else one positive and the other two negative. For, assuming it to be so, we have

$$0 = ABC - F'G'H',$$

that is,

$$0 = A(BC - F'^2) - F'(G'H' - AF'),$$
$$= K'(Aa + F'f), = K'(abc + fgh);$$

which is right in virtue of the relation $K' = 0$. Taking the signs all positive, we find for $(x^2, y^2, z^2, yz, zx, xy)$ the values (A, B, C, F', G', H'), giving two points of intersection

$$\left(\sqrt{A}, \frac{H'}{\sqrt{A}}, \frac{G'}{\sqrt{A}}\right) \text{ and } \left(-\sqrt{A}, -\frac{H'}{\sqrt{A}}, -\frac{G'}{\sqrt{A}}\right).$$

Taking the signs one positive and the other two negative, say

$$\sqrt{BC} = F', \quad \sqrt{CA} = -G', \quad \sqrt{AB} = -H',$$

we find for $(x^2, y^2, z^2, yz, zx, xy)$ the values

$$\left(0, \frac{Cc}{b}, \frac{Bb}{c}, F', 0, 0\right),$$

viz., we have thus two intersections

$$\left(0, \sqrt{\frac{Cc}{b}}, F'\sqrt{\frac{b}{Cc}}\right), \quad \left(-G'\sqrt{\frac{Cc}{b}}, -F'\sqrt{\frac{b}{Cc}}\right),$$

and the other combinations of signs give the remaining two pairs of intersections

$$\left(G'\sqrt{\frac{c}{Aa}}, 0, \sqrt{\frac{Aa}{c}}\right), \quad \left(-G'\sqrt{\frac{c}{Aa}}, 0, -\sqrt{\frac{Aa}{c}}\right),$$

and

$$\left(\sqrt{\frac{Bb}{a}}, H'\sqrt{\frac{a}{Bb}}, 0\right), \quad \left[-\sqrt{\frac{Bb}{a}}, -H'\sqrt{\frac{a}{Bb}}, 0\right].$$

But the most convenient statement of the result is that the values of $(ax^2, by^2, cz^2, yz, zx, xy)$ for the four pairs of points respectively are

$$(aA, bB, cC, F', G', H'),$$
$$(0, cC, bB, F', 0, 0),$$
$$(cC, 0, aA, 0, G', 0),$$
$$(bB, aA, 0, 0, 0, H');$$

there is no difficulty in substituting these values in the original equations, and in verifying that the equations are in each case satisfied.

On the Solution of certain Partial Differential Equations of the Second Order having more than Two Independent Variables.

By H. W. Lloyd Tanner, M.A., Professor of Mathematics and Physics, Royal Agricultural College, Cirencester.

[*Read Dec. 9th*, 1875.]

The equations in question are included in the form

$$\sum_{i=1}^{i=n}\sum_{j=1}^{j=n} V_{ij}\frac{d^2z}{dx_i dx_j} + V_0 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where V_{ij}, V_0 are functions of $x_1 \ldots x_n$, z, $p_1\left(\equiv\frac{dz}{dx_1}\right)$, $\ldots$ $p_n\left(\equiv\frac{dz}{dx_n}\right)$.

"Suppose that there exists a solution

$$F\{x_1, x_2 \ldots x_n, z, \phi(u_1, u_2 \ldots u_{n-1}) \ldots\} = 0$$

involving an arbitrary function ϕ of the $n-1$ quantities $u_1, u_2, \ldots u_{n-1}$, functions of $z, x_1, x_2 \ldots x_n$, whose values are independent of the form of ϕ. It is obvious that the arguments $u_1, u_2 \ldots u_{n-1}$ (although they may without loss of generality be regarded as determinate functions of $z_1, x_1, x_2, \ldots x_n$) are not of necessity determinate functions of $z, x_1 \ldots x_n$; for, taking $v_1, v_2, \ldots v_{n-1}$ any particular values of $u_1, u_2, \ldots u_{n-1}$, then the general values are $u_1, u_2 \ldots u_{n-1}$, each of them equal to an arbitrary function of $v_1, v_2 \ldots v_{n-1}$; and it hence follows that $u_1, u_2 \ldots u_{n-1}$ are each of them a solution of one and the same partial differential equation of the first order, viz., this is the equation

$$\begin{vmatrix} \left(\frac{du}{dx_1}\right), & \left(\frac{du}{dx_2}\right), & \ldots & \left(\frac{du}{dx_n}\right) \\ \left(\frac{dv_1}{dx_1}\right), & \left(\frac{dv_1}{dx_2}\right), & \ldots & \left(\frac{dv_1}{dx_n}\right) \\ \ldots & \ldots & & \ldots \\ \left(\frac{dv_{n-1}}{dx_1}\right), & \left(\frac{dv_{n-1}}{dx_2}\right), & \ldots & \left(\frac{dv_{n-1}}{dx_n}\right) \end{vmatrix} = 0,$$

where $v_1, \ldots v_{n-1}$ are regarded as given functions of $z, x_1, \ldots x_n$, and $\left(\frac{dv_1}{dx_1}\right)$ means $\frac{dv_1}{dx_1} + p_1\frac{dv_1}{dz}$, &c. ....

"We propose in the first instance to investigate the conditions which must be satisfied in order that (1) may have a solution of the form above referred to; and when the conditions are satisfied, to determine the solution. It will appear as a result that in certain cases there will be two sets of arguments—say $u_1, u_2, \ldots u_{n-1}$ the arguments of an arbitrary function φ, and $\bar{u}_1, \bar{u}_2, \ldots \bar{u}_{n-1}$ the arguments of a second arbitrary function $\bar{\varphi}$, where in like manner $\bar{u}_1, \bar{u}_2, \ldots \bar{u}_{n-1}$ are not of

necessity determinate functions of $z, x_1, \ldots x_n$, but are the solutions of one and the same partial differential equation obtained by consideration of a set of particular solutions $\overline{v}_1, \overline{v}_2, \ldots \overline{v}_{n-1}$; or that there will be a solution

$$F\{x_1, x_2, \ldots x_n, z, \phi(u_1, u_2, \ldots u_{n-1}), \overline{\phi}(\overline{u}_1, \overline{u}_2, \ldots \overline{u}_{n-1})\} = 0,$$

or, it may be, involving also the integrals of these arbitrary functions in regard to each or any of the arguments.

"The arguments $u_1, \ldots u_{n-1}$ and $\overline{u}_1, \overline{u}_2, \ldots \overline{u}_{n-1}$ may without loss of generality be regarded as determinate functions of $z, x_1, x_2, \ldots x_n$; and it is convenient so to regard them. Say therefore that there are $2n-2$ arguments $u_1 \ldots u_{n-1}$ and $\overline{u}_1 \ldots \overline{u}_{n-1}$, each of them a determinate function of $z, x_1, x_2, \ldots x_n$." *

Three cases arise for discussion.

1. If $\overline{n-1}$ of these arguments be independent, the given equation can be transformed to an ordinary equation of the second order. The arbitrary constants in the solution of this are to be replaced by arbitrary functions, of each of which the arguments will be the $\overline{n-1}$ independent quantities. In this case we have for all values of i, j,

$$V_{ii}V_{jj} - V_{ij}^2 = 0.$$

2. If n of the arguments be independent, the equation is always reducible to the form

$$\sum_{i=1}^{i=n-1} A_i \left(\frac{d^2\zeta}{d\xi_i d\eta_1} + P\frac{d\zeta}{d\xi_i} \right) + Q\frac{d\zeta}{d\eta_1} + Z' = 0,$$

where ξ_i, η_1 are the n independent arguments. In this equation A_i, P, Q are functions of $\xi_i \ldots \eta_1$ only, and the coefficients A_i do not involve η_1.

If Z' be linear in ζ ($\equiv Z\zeta + W$ say), and the conditions

$$\Sigma A_i \frac{dP}{d\xi_i} = Z - PQ = \frac{dQ}{d\eta_1}$$

be satisfied, the solution contains the arbitrary functions only.

If Z' be linear in ζ, but these conditions are not fulfilled, the solution is of the form

$$\zeta = \zeta_0 + \sum_{j=0}^{j=k} X_j \left(\Sigma_i A_i \frac{d}{d\xi_i} \right)^{k-j} \Phi(\xi_1 \ldots \xi_{n-1}) + \sum_{j=0}^{j=l} H_j \left(\frac{d}{d\eta_1} \right)^{l-j} \Psi(\eta_1, \eta_2 \ldots \eta_{n-1});$$

and the quantities X_j, H_j, $\eta_2 \ldots \eta_{n-1}$ are determined by linear equations of the first order.

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* I have to thank Prof. Cayley for this statement of the question proposed for investigation in the present paper.

If Z' be not linear in ζ, the solution, if it exist, must involve integrals of the arbitrary functions (which may be replaced by derivatives if the solution is finite). This case is not discussed in the present paper.

3. The case in which $n+1$ of the arguments are independent is also omitted from the present investigation. It is noticeable that this case has no analogue in the case of equations with two independent variables.

No other cases can arise, since the $\overline{n-1}$ arguments of either function must be different. Hence there cannot be less than $\overline{n-1}$ independent arguments. Neither can there be more than $\overline{n+1}$ independent arguments, since these arguments are by hypothesis functions of only $n+1$ independent quantities $x_1 \ldots x_n, z$.

When r of the arguments of one function are identical with r of the other, (1) can be transformed to an equation having only $\overline{n-r}$ independent variables.

The paper concludes with a note on the application of a similar method to equations of an order higher than the second.

Determination of the Arguments.

1. Let one of the arguments of one of the arbitrary functions (ϕ say) in the solution be u. We may assume that none of the derivatives of ϕ with respect to u occur in the solution. For, supposing $\frac{d^r\phi}{du^r}$ to be the highest derivative, we may replace it by an arbitrary function ψ, so that any lower derivative $\frac{d^s\phi}{du^s}$ would be represented by the indefinite integral $\frac{d^{s-r}}{du^{s-r}}\psi(u)$. The solution may therefore be written in the form

$$\mathrm{F}\{x_1, x_2 \ldots x_n, z, \phi(u, \ldots), \ldots\} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

where F involves none of the derivatives of ϕ with respect to u.

Differentiate (2) with respect to x_i, x_j in succession, and retain only the terms which involve the derivative $\frac{d^2\phi}{du^2}$, either directly, or implicitly in $\frac{d^2z}{dx_i\,dx_j}$. Thus

$$\frac{d^2z}{dx_i\,dx_j}\left\{\frac{d\mathrm{F}}{dz} + \frac{d\mathrm{F}}{d\phi}\cdot\frac{d\phi}{du}\cdot\frac{du}{dz} + \&c.\ldots\right\} + \ldots + \frac{d\mathrm{F}}{d\phi}\cdot\frac{d^2\phi}{du^2}\left(\frac{du}{dx_i}\right)\left(\frac{du}{dx_j}\right) + \ldots = 0,$$

where $\left(\frac{du}{dx_i}\right) \equiv \frac{du}{dx_i} + p_i\frac{du}{dz}$.

Multiply by V_{ij}, and take the sum of all the equations formed by

giving to i, j all values from 1 to n inclusive. Since (2) is a solution of (1) this sum may be written in the form

$$-V_0\left\{\frac{dF}{dz}+\frac{dF}{d\phi}\cdot\frac{d\phi}{du}\cdot\frac{du}{dz}+\dots\right\}+\dots+\frac{dF}{d\phi}\cdot\frac{d^2\phi}{du^2}\cdot\Sigma_i\Sigma_j V_{ij}\left(\frac{du}{dx_i}\right)\left(\frac{du}{dx_j}\right)$$
$$+\ \&c.\ \dots\dots=0.$$

From this equation all the second derivatives of z have disappeared, and since, by (2), $p_1 \dots p_n$ do not involve $\frac{d^2\phi}{du^2}$, the only term in which this arbitrary function occurs is the second of those written above. The coefficient of $\frac{d^2\phi}{du^2}$ in this term must therefore vanish; that is, since

$$\left.\begin{array}{l}\frac{dF}{d\phi} \text{ is not equal to } 0, \quad \Sigma_i\Sigma_j V_{ij}\left(\frac{du}{dx_i}\right)\left(\frac{du}{dx_j}\right)=0,\\ \text{or}\quad V_{11}\left(\frac{du}{dx_1}\right)^2+V_{22}\left(\frac{du}{dx_2}\right)^2+\dots+2V_{12}\left(\frac{du}{dx_1}\right)\left(\frac{du}{dx_2}\right)+\dots=0\end{array}\right\}\quad\dots(3),$$

an equation for the determination of u.

2. Now if $u_1, u_2 \dots$ are the arguments of ϕ, viz., if they are particular values of u, the general value of u is

$$u=\phi(u_1, u_2 \dots),$$

where ϕ is arbitrary. Thus u satisfies an equation of the form

$$\left.A_1\left(\frac{du}{dx_1}\right)+A_2\left(\frac{du}{dx_2}\right)+\dots=0,\right.$$

namely, the equation

$$\left.\begin{vmatrix} \left(\frac{du}{dx_1}\right), & \left(\frac{du}{dx_2}\right), & \dots & \left(\frac{du}{dx_n}\right)\\ \frac{du_1}{dx_1}+p_1\frac{du_1}{dz}, & \frac{du_1}{dx}+p_2\frac{du_1}{dz}, & \dots & \dots\\ \dots & \dots & \dots & \dots\\ \frac{du_{n-1}}{dx_1}+p_1\frac{du_{n-1}}{dz}, & \dots & \dots & \dots \end{vmatrix}=0.\right\}\quad\dots(4).$$

Hence the expression on the left of (3) breaks up into two factors linear with respect to $\left(\frac{du}{dx_1}\right)$, &c., each of which gives a set of arguments.

It thus appears that the $\frac{n(n+1)}{1.2}-1$ ratios $\frac{V_{ii}}{V_{11}}$, $\frac{V_{ij}}{V_{11}}$ are expressible in terms of $2n-2$ ratios $\frac{A_2}{A_1}$, $\frac{A_i}{A_1}$ ..., $\frac{B_2}{B_1}$, $\frac{B_i}{B_1}$ ..., and are therefore connected by $\frac{n(n+1)}{1.2}-1-(2n-2)$ or $\frac{(n-1)(n-2)}{1.2}$ equations

of condition. These are evidently the $\frac{(n-1)(n-2)}{1.2}$ different equations that can be formed by giving to i, j all values from 2 to n inclusive in the equation

$$\begin{vmatrix} V_{11}, & V_{1i}, & V_{1j} \\ V_{i1}, & V_{ii}, & V_{ij} \\ V_{j1}, & V_{ji}, & V_{jj} \end{vmatrix} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

For this is the condition that

$$V_{11} = \lambda A_1 B_1, \quad V_{ii} = \lambda A_i B_i, \quad V_{jj} = \lambda A_j B_j,$$

$$2V_{ij} = \lambda(A_i B_j + A_j B_i), \quad 2V_{j1} = \lambda(A_j B_1 + A_1 B_j), \quad 2V_{i1} = \lambda(A_1 B_i + A_i B_1);$$

and if this be true for all values of i, j, the expression on the left of (3) is identically equal to

$$\lambda \left\{ A_1 \left(\frac{du}{dx_1}\right) + \ldots \right\} \left\{ B_1 \left(\frac{du}{dx_1}\right) + \ldots \right\}.$$

3. Let us now examine the form of A, &c. in the factors of (3). From (4) it is clear that the coefficient of $\left(\frac{du}{dx_i}\right)$ does not contain p_i. Moreover, it is linear with respect to p_1, p_2, &c. For, if we write down the coefficient of such a product as $p_1 p_2$ in the coefficient of $\left(\frac{du}{dx_i}\right)$, we find it to be a determinant having the column $\frac{du_1}{dz}$, $\frac{du_2}{dz}$, &c. repeated; viz, that it is identically zero. Hence we may write

$$A_i = a_i + a_{i1} p_1 + \ldots + a_{ij} p_j + \ldots \quad \ldots\ldots\ldots\ldots\ldots (6),$$

and

$$a_{ii} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7).$$

The $n^2 - 1$ ratios between $a_i \ldots$, $a_{ii} \ldots$, $a_{ij} \ldots$, $a_{ji} \ldots$ are, by (4), expressible in terms of $n-1$ quantities $u_1 \ldots u_{n-1}$, and are therefore connected by $n(n-1)$ equations of condition. These are, of course, implied in (4), and might be obtained thence. We prefer to adopt a different course.

Since u_1, &c. are functions of $x_1 \ldots x_n$, z only, they must satisfy the equation for u independently of the values of $p_1 \ldots p_n$; that is, the coefficients of the powers and products of $p_1 \ldots p_n$ must separately vanish.

The product $p_i p_j$ only occurs in the terms $A_i \left(\frac{du}{dx_i}\right)$, $A_j \left(\frac{du}{dx_j}\right)$, and expressing that its coefficient in (4) vanishes we get the equation

$$a_{ij} + a_{ji} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8).$$

The conditions that the coefficient of p_i and the part independent

of p_i, &c. vanish, are

$$\left.\begin{aligned} a_1\frac{du}{dx_1}+\ldots\ldots+a_n\frac{du}{dx_n}&=0\\ \ldots\ldots\qquad&\qquad\ldots\ldots\\ (\mathrm{I}\equiv)\ a_i\frac{du}{dz}+\ldots\ldots+a_{in}\frac{du}{dx_n}&=0\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9).$$

The equations (9) must be satisfied by the $\overline{n-1}$ different integrals $w_1 \ldots w_{n-1}$. But the system (9) only contains $\overline{n+1}$ independent variables $(x_1 \ldots x_n, z)$. Hence it must be equivalent to two equations only. Suppose these to be $\mathrm{I}=0$, $\mathrm{J}=0$; then any other equation $\mathrm{K}=0$ must be of the form $\mathrm{K}\equiv\lambda\mathrm{I}+\mu\mathrm{J}$.

Expressing that the coefficients of $\frac{du}{dx_i}$, $\frac{du}{dx_j}$, $\frac{du}{dx_l}$ on each side of this identity are equal, we get the equations

$$a_{ki}=\mu a_{ji},\qquad a_{kj}=\lambda a_{ij},\qquad a_{kl}=\lambda a_{il}+\mu a_{jl};$$

whence, eliminating λ, μ, and reducing by (8),

$$a_{ij}a_{kl}+a_{ik}a_{lj}+a_{il}a_{jk}=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10).$$

From this we get $(n-1)^2$ different conditions by giving to k, l all values from 0 to n inclusive, except i, j. These include (8) as a special case, viz., when $k=l$.

The conditions that $\mathrm{I}=0$, $\mathrm{J}=0$ should have $\overline{n-1}$ common integrals have been given by Boole (Supplementary Vol., chap. XXV.). If we write

$$\Delta_i\equiv\frac{d}{dx_j}+\frac{a_i}{a_{ij}}\cdot\frac{d}{dz}+\ldots+\frac{a_{ik}}{a_{ij}}\cdot\frac{d}{dx_k}+\ldots,$$

$$\Delta_j\equiv\frac{d}{dx_i}+\frac{a_j}{a_{ji}}\cdot\frac{d}{dz}+\ldots+\frac{a_{jk}}{a_{ji}}\cdot\frac{d}{dx_k}+\ldots,$$

the conditions are

$$\left.\begin{aligned}\Delta_i\frac{a_j}{a_{ji}}&=\Delta_j\frac{a_i}{a_{ij}}\\ \ldots\ldots\qquad&\qquad\ldots\ldots\\ \Delta_i\frac{a_{jk}}{a_{ji}}&=\Delta_j\frac{a_{ik}}{a_{ij}}\\ \ldots\ldots\qquad&\qquad\ldots\ldots\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(11).$$

The $(n-1)$ conditions (11) and the $(n-1)^2$ conditions (10) together make up the $n(n-1)$ conditions which must be satisfied by the coefficients a. The sufficiency of the conditions is proved by the fact that they insure, first, that two independent equations of the system (9) shall have $n-1$ common integrals, which are the arguments required; second, that these integrals shall satisfy all the other equations (9).

4. *Example:*

$$(p_2-p_3)\left(\frac{d^2z}{dx_1^2}-\frac{d^2z}{dx_2dx_3}\right)+(p_3-p_1)\left(\frac{d^2z}{dx_2^2}-\frac{d^2z}{dx_3dx_1}\right)$$
$$+(p_1-p_2)\left(\frac{d^2z}{dx_3^2}-\frac{d^2z}{dx_1dx_2}\right)+\mathrm{V}=0.$$

This equation satisfies the single condition (5), and the factors of (3) are found to be $\left(\frac{du}{dx_1}\right)+\left(\frac{du}{dx_2}\right)+\left(\frac{du}{dx_3}\right)=0,$

$$(p_2-p_3)\left(\frac{du}{dx_1}\right)+(p_3-p_1)\left(\frac{du}{dx_2}\right)+(p_1-p_2)\left(\frac{du}{dx_1}\right)=0.$$

The first of these gives us

$$\frac{du}{dx_1}+\frac{du}{dx_2}+\frac{du}{dx_3}=0,$$
$$\frac{du}{dz}=0,$$

simultaneously, whence $u=\phi(x_2-x_3,\ x_3-x_1).$

The second gives $a_1=a_2=a_3=0,$

$$a_{23}=1=-a_{32},\quad a_{31}=1=-a_{13},\quad a_{12}=1=-a_{21};$$

and the equations (9) reduce to

$$\frac{du}{dx_1}=\frac{du}{dx_2}=\frac{du}{dx_3};$$

whence $u=\psi(x_1+x_2+x_3,\ z).$

In fact, the solution of the given equation, when $\mathrm{V}=0$, is

$$\phi(x_2-x_3,\ x_3-x_1)+\psi(x_1+x_2+x_3,\ z)=0.$$

Transformation of the Variables.

5. Suppose that one of the arguments is identical with one of the independent variables, say x_1, so that x_1 satisfies the equations (9). This requires that the coefficient of $\frac{du}{dx_1}$ should vanish in each of the equations, or $a_1=a_{12}=a_{13}=\ldots=a_{1n}=0.$

Also, by (8), $a_{21}=a_{31}=\ldots=a_{n1}=0;$

that is, $\mathrm{A}_1=0$, and none of the coefficients A involve p_1.

So if r of the arguments be $x_1,\ldots x_r$ the coefficients $\mathrm{A}_1\ldots\mathrm{A}_r$ vanish, and the remaining coefficients do not involve $p_1\ldots p_r$.

By using these results we can simplify the equation (1) without undertaking the laborious task of actually transforming the variables.

6. If (3) is a perfect square—viz., if for all values of i, j,

$$V_{ii}V_{jj} - V_{ij}^2 = 0,$$

the two sets of arguments become identical. Let them be represented by $\xi_1 \ldots \xi_{n-1}$. Take two functions of $x_1 \ldots x_n, z$ which are independent of these and each other. Represent them by ξ_n, ζ. Now transform the equation, taking $\xi_1 \ldots, \zeta$ as variables instead of $x_1 \ldots z$. By Art. 5 each of the factors into which (3) splits up becomes simply

$$\left(\frac{du}{dx_n}\right) = 0.$$

Thus all the coefficients V of the transformed equation vanish except V_{nn}, and the transformed equation is therefore of the form

$$\frac{d^2\zeta}{d\xi_n^2} + V_0 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12).$$

7. If (3) be not a perfect square, so that the two sets of arguments are not identical, let these arguments be represented by $\xi_1 \ldots \xi_{n-1}$, $\eta_1 \ldots \eta_{n-1}$. For the sake of generality we may assume that r of these arguments are identical with r of the other set; *e. g.*,

$$\xi_1 = \eta_1, \ \ldots\ldots \ \xi_r = \eta_r.$$

Take for new independent variables $\xi_1 \ldots \xi_{n-1}, \eta$, the last being any one of the arguments $\eta_{r+1} \ldots \eta_{n-1}$.

For dependent variable take ζ such that $\xi_1 \ldots \xi_{n-1}, \eta, \zeta$ considered as functions of $x_1 \ldots x_n, z$ may be mutually independent.

In this case the equations into which (3) splits up are of the form

$$\left(\frac{du}{d\eta}\right) = 0,$$

$$A_{r+1}\left(\frac{du}{d\xi_{r+1}}\right) + \ldots + A_{n-1}\left(\frac{du}{d\xi_{n-1}}\right) = 0,$$

where $A_{r+1} \ldots A_{n-1}$ do not involve $\frac{d\zeta}{d\xi_1}, \ \ldots \ \frac{d\zeta}{d\xi_r}$.

The transformed equation is therefore of the form

$$A_{r+1}\frac{d^2\zeta}{d\xi_{r+1}d\eta} + \ldots + A_{n-1}\frac{d^2\zeta}{d\xi_{n-1}d\eta} + V_0 = 0 \quad \ldots\ldots\ldots\ldots (13).$$

It is to be noted that there is a considerable range of choice in the new independent variables. These may be either $\xi_1 \ldots \xi_{n-1}$ with any one of the arguments $\eta_{r+1} \ldots \eta_{n-1}$, or they may be $\eta_1 \ldots \eta_{n-1}$ with any one of the arguments $\xi_{r+1} \ldots \xi_{n-1}$.

The two equations (12), (13) are together equivalent to (1) limited by the conditions of Arts. 2, 3. We shall therefore examine the form of V_0 as it occurs in these equations instead of in the general form.

On the Form of V_0.

8. Consider first the equation

$$\frac{d^2\zeta}{d\xi_n^2} + V_0 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12).$$

If this be capable of exact solution in the form (2), V_0 cannot involve $\frac{d\zeta}{d\xi_1} \ldots \frac{d\zeta}{d\xi_{n-1}}$. For $\frac{d\zeta}{d\xi}$, for instance, involves the arbitrary function $\frac{d\phi}{d\xi_1}$, which does not occur in any other part of (12). Thus, (2) cannot be a solution of (12) if V_0 contain $\frac{d\zeta}{d\xi_1}$, unless some condition is imposed upon the form of ϕ; *i. e.*, unless ϕ ceases to be arbitrary. Hence V_0 is a function of $\frac{d\zeta}{d\xi_n}$ and the variables $\xi_1 \ldots \xi_n . \zeta$ only, or (12) is an ordinary differential equation of the second order. The solution of this equation involves two arbitrary constants; that is, two arbitrary quantities whose values do not change when ξ_n changes; that is, two arbitrary functions of $\xi_1 \ldots \xi_{n-1}$. The solution of the original equation is, of course, obtained from this by writing for $\xi_1 \ldots \xi_n . \zeta$ their values in terms of $x_1 \ldots x_n, z$.

9. In the second form of equation, viz.,

$$A_{r+1}\frac{d^2\zeta}{d\xi_{r+1}d\eta} + \ldots + A_{n-1}\frac{d^2\zeta}{d\xi_{n-1}d\eta} + V_0 = 0 \quad \ldots\ldots\ldots\ldots (13),$$

$A_{r+1}, \ldots A_{n-1}$ do not involve $\frac{d\zeta}{d\xi_1} \ldots \frac{d\zeta}{d\xi_r}$. Hence, by precisely the same reasoning as in the last article, V_0 cannot involve these quantities, and the equation (13) is therefore reduced to an equation with $\overline{n-r}$ independent variables. We shall discuss the equation under the supposition $r=0$, since, by the above, (13) is, in fact, of the form of an equation in which this condition is fulfilled. We write the equation, then, in the form

$$A_1\frac{d^2\zeta}{d\xi_1 d\eta_1} + \ldots + A_{n-1}\frac{d^2\zeta}{d\xi_{n-1}d\eta_1} + V_0 = 0 \quad \ldots\ldots\ldots\ldots (14).$$

The arguments are

$$\xi_1 \ldots \xi_{n-1},$$
$$\eta_1 \ldots \eta_{n-1},$$

where $\eta_1 \ldots \eta_{n-1}$ are $\overline{n-1}$ independent integrals of

$$\sum_{i=1}^{i=n-1} A_i \frac{d\eta}{d\xi_i} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (15).$$

Two cases arise. If of the $2n-2$ arguments n only are mutually independent when considered as functions of $x_1 \ldots x_n, z$, then none of

the quantities $\eta_2 \dots \eta_{n-1}$ involve the new dependent variable ζ, for this, considered as function of $x_1 \dots x_n, z$, is independent of $\xi_1, \dots \xi_{n-1}, \eta$. If, however, $n+1$ of the arguments are mutually independent, then $\eta_2 \dots \eta_{n-1}$ or some of them must involve ζ. We omit the discussion of this case in the present paper, and confine our attention to the first alone; viz., we assume that in the transformed equation the arguments do not involve the dependent variable, and that accordingly the coefficients A_i do not involve ζ or any of its first derivatives $\frac{d\zeta}{d\xi_i}, \dots \frac{d\zeta}{d\eta_1}$.

10. Under these circumstances ζ must occur explicitly in F in (2); thus we may solve for ζ and write the solution in the more convenient form

$$\zeta = \mathrm{F}\{\xi \dots \xi_i \dots \eta_1,\ \phi(\xi \dots),\ \psi(\dots \eta \dots),\ \dots\} \quad \dots\dots\dots\dots(16).$$

Form from this the derivatives $\frac{d\zeta}{d\xi_i}, \frac{d\zeta}{d\eta}, \frac{d^2\zeta}{d\xi_i d\eta}$, and retain only the terms which involve the derivatives of ϕ, ψ. Thus,

$$\left.\begin{aligned} \frac{d\zeta}{d\xi_i} &= \frac{d\mathrm{F}}{d\xi_i} + \frac{d\mathrm{F}}{d\phi}\cdot\frac{d\phi}{d\xi_i} + \dots + \frac{d\mathrm{F}}{d\psi}\left(\frac{d\psi}{d\eta_1}\cdot\frac{d\eta_2}{d\xi_i} + \dots\right) + \dots \\ \frac{d\zeta}{d\eta_1} &= \frac{d\mathrm{F}}{d\eta_1} + \frac{d\mathrm{F}}{d\psi}\left(\frac{d\psi}{d\eta_1} + \frac{d\psi}{d\eta_2}\cdot\frac{d\eta_2}{d\eta_1} + \dots\right) + \dots \end{aligned}\right\} \dots\dots(17);$$

$$\begin{aligned} \frac{d^2\zeta}{d\xi_i d\eta_1} = {} & \frac{d^2\mathrm{F}}{d\xi_i d\eta_1} + \frac{d^2\mathrm{F}}{d\xi_i d\psi}\left\{\frac{d\psi}{d\eta_1} + \frac{d\psi}{d\eta_2}\cdot\frac{d\eta_2}{d\eta_1} + \dots\right\} + \dots \\ & + \frac{d^2\mathrm{F}}{d\eta_1 d\phi}\cdot\frac{d\phi}{d\xi_i} + \frac{d^2\mathrm{F}}{d\psi d\phi}\cdot\frac{d\phi}{d\xi_i}\left\{\frac{d\psi}{d\eta_1} + \frac{d\psi}{d\eta_2}\cdot\frac{d\eta_2}{d\eta_1} + \dots\right\} + \dots \\ & + \frac{d^2\mathrm{F}}{d\eta_1 d\psi}\left\{\frac{d\psi}{d\eta_2}\cdot\frac{d\eta_2}{d\xi_i} + \dots\right\} \\ & + \frac{d^2\mathrm{F}}{d\psi^2}\left\{\frac{d\psi}{d\eta_2}\cdot\frac{d\eta_2}{d\xi_i} + \dots\right\}\left\{\frac{d\psi}{d\eta_1} + \frac{d\psi}{d\eta_2}\cdot\frac{d\eta_2}{d\eta_1} + \dots\right\} + \dots \\ & + \frac{d\mathrm{F}}{d\psi}\left\{\frac{d^2\psi}{d\eta_1 d\eta_2}\cdot\frac{d\eta_2}{d\xi_i} + \dots + \frac{d\psi}{d\eta_2}\cdot\frac{d^2\eta_2}{d\eta_1 d\xi_i} + \dots\right\} + \dots \end{aligned}$$

A remark as to the completeness of this value of $\frac{d^2\zeta}{d\xi_i d\eta_1}$ is necessary. Since two differentiations of ζ are performed, it is clear that derivatives of ϕ and ψ of the first order may arise from integrals of these functions in ζ; and as a matter of fact such derivatives will occur in $\frac{d^2\zeta}{d\xi_i d\eta_1}$, but will disappear from the equation (14). For this equation, as far as terms of the second order are concerned, may be written in the form

$$\Sigma_i \mathrm{A}_i \frac{d}{d\xi_i}\cdot\frac{d}{d\eta_1}\cdot\zeta = 0.$$

Now of these two operators $\frac{d}{d\eta_1}$ does not affect ϕ, for the arguments of this function are independent of η_1, and $\Sigma A_i \frac{d}{d\xi_i}$ does not affect ψ, since each of the arguments of this function satisfies the equation (15). Hence neither of the arbitrary functions will be differentiated twice, and their derivatives of the first order can only arise from the part of ζ involving ϕ, ψ.

An illustration occurs which may throw some light on this point. In the value of $\frac{d^2\zeta}{d\xi_i d\eta_1}$ we have the function $\frac{d^2\psi}{d\eta_1 d\eta_2}$. By what has been stated above, this function must disappear from (14), and we find its coefficient in (14) to be $\frac{dF}{d\psi} \cdot \Sigma A_i \frac{d\eta_2}{d\xi_i}$,

of which the second factor vanishes, because η_2 is a solution of (15).

In virtue of the same condition, other parts of the value of $\frac{d^2\zeta}{d\xi_i d\eta}$ will disappear on substitution in (14). If we retain only those terms which involve the derivatives and which will be represented in (14) we obtain the simplified equation

$$\frac{d^2\zeta}{d\xi_i d\eta_1} = \frac{d^2F}{d\phi\, d\psi} \cdot \frac{d\phi}{d\xi_i}\left(\frac{d\psi}{d\eta_1}\right) + \frac{d^2F}{d\eta_1 d\phi} \cdot \frac{d\phi}{d\xi_i} + \frac{d^2F}{d\xi_i d\psi}\left(\frac{d\psi}{d\eta_1}\right) + \frac{dF}{d\psi} \cdot \sum_{j=2}^{j=n-1} \frac{d\psi}{d\eta_j} \cdot \frac{d^2\eta_j}{d\eta_1 d\xi_i} + \ldots \quad \ldots\ldots (18),$$

where $$\left(\frac{d\psi}{d\eta_1}\right) \equiv \frac{d\psi}{d\eta_1} + \frac{d\psi}{d\eta_2} \cdot \frac{d\eta_2}{d\eta_1} + \ldots.$$

In order that ζ should satisfy (14) it is necessary that, when we substitute (17) and (18) in that equation, the coefficient of each power and product of the arbitrary functions should separately vanish. The conditions hence obtained will serve in great measure to determine the form of V_0 in (14), and of F in (16).

11. In the first place, V_0 must not involve powers or products of the arbitrary functions other than those which occur in (18). Hence the most general form of V_0 is

$$\Sigma_i B_i \frac{d\zeta}{d\xi_i} \cdot \frac{d\zeta}{d\eta_1} + \Sigma_i P_i \frac{d\zeta}{d\xi_i} + Q \frac{d\zeta}{d\eta_1} + Z',$$

where B_i, P_i, Q, Z′ do not involve $\frac{d\zeta}{d\xi_i}$ or $\frac{d\zeta}{d\eta_1}$. Now, equating to zero the coefficient of $\frac{d\phi}{d\xi_i}\left(\frac{d\psi}{d\eta_1}\right)$ in (14), we get

$$A_i \frac{d^2F}{d\phi\, d\psi} + B_i \frac{dF}{d\psi} \cdot \frac{dF}{d\phi} = 0 \ldots\ldots\ldots\ldots\ldots (19).$$

Hence the ratio $A_i : B_i$ is independent of i; viz., (14) may be written

$$\Sigma_i A_i \left(\frac{d^2\zeta}{d\xi_i d\eta_1} + k \frac{d\zeta}{d\xi_i} \cdot \frac{d\zeta}{d\eta_1} \right) + \ldots = 0.$$

We shall now show that we can make k vanish by a change of the dependent variable only. Take as new dependent variable

$$\zeta' \equiv f(\zeta, \xi, \ldots \xi_{n-1}, \eta_1);$$

whence

$$\frac{d\zeta'}{d\xi_i} = \frac{df}{d\zeta} \cdot \frac{d\zeta}{d\xi_i} + \frac{df}{d\xi_i},$$

$$\frac{d\zeta'}{d\eta_1} = \frac{df}{d\zeta} \cdot \frac{d\zeta}{d\eta_1} + \frac{df}{d\eta_1},$$

$$\frac{d^2\zeta'}{d\xi_i d\eta_1} = \frac{df}{d\zeta} \cdot \frac{d^2\zeta}{d\xi_i d\eta_1} + \frac{d^2f}{d\zeta^2} \cdot \frac{d\zeta}{d\xi_i} \cdot \frac{d\zeta}{d\eta_1} + \frac{d^2f}{d\xi_i d\zeta} \cdot \frac{d\zeta}{d\eta_1} + \frac{d^2f}{d\eta_1 d\zeta} \cdot \frac{d\zeta}{d\xi_1} + \frac{d^2f}{d\xi_i d\eta_1}.$$

Hence the expression

$$A_i \frac{d^2\zeta'}{d\xi_i d\eta_1} + P_i' \frac{d\zeta'}{d\xi_i} + Q_i' \frac{d\zeta'}{d\eta_1} + Z_i''$$

is equivalent to the expression

$$A_i \frac{d^2\zeta}{d\xi_i d\eta_1} + A_i k \frac{d\zeta}{d\xi_1} \cdot \frac{d\zeta}{d\eta_1} + P_i \frac{d\zeta}{d\xi_i} + Q_i \frac{d\zeta}{d\eta_1} + Z_i',$$

provided we have

$$\frac{d^2f}{d\zeta^2} = k \frac{df}{d\zeta},$$

$$\frac{d^2f}{d\eta_1 d\zeta} + P_i' \frac{df}{d\zeta} = P_i \frac{df}{d\zeta},$$

$$\frac{d^2f}{d\xi_i d\zeta} + Q_i' \frac{df}{d\zeta} = Q_i \frac{df}{d\zeta},$$

$$\frac{d^2f}{d\xi_i d\eta_1} + P_i' \frac{df}{d\xi_i} + Q_i' \frac{df}{d\eta_1} + Z_i'' = Z_i' \frac{df}{d\zeta}.$$

The first of these determines ζ'; viz.,

$$\zeta' \equiv f = \int e^{\int k\, dz}\, dz,$$

while the others determine the values of P_i', Q_i', Z_i'' when P_i, Q_i, Z_i' are given.

Now this value of ζ' is independent of i, and therefore, if (14) be changed to its equivalent with ζ' as dependent variable, the resulting equation will be linear with respect to $\frac{d\zeta'}{d\xi_i}$, $\frac{d\zeta'}{d\eta_1}$. We shall therefore assume for the future that $B_i = 0$, since (14) may always be reduced

to an equation in which this is true, if it be soluble in the assumed form.

On this assumption it follows from (19) that $\frac{d^2F}{d\phi\, d\psi} = 0$; viz., in (16) the two arbitrary functions are separated.

12. From the coefficient of $\frac{d\phi}{d\xi_i}$ we learn that

$$A_i \frac{d^2F}{d\eta_1 d\phi} + P_i \frac{dF}{d\phi} = 0,$$

so that the ratio $A_i : P_i$ is the same for all values of i; and we may write (14) in the form

$$\Sigma A_i \left(\frac{d^2\zeta}{d\xi_i d\eta_1} + P \frac{d\zeta}{d\xi_i} \right) + Q \frac{d\zeta}{d\eta_1} + Z' = 0 \quad \ldots\ldots\ldots (20).$$

The conditions for all values of i then become

$$\frac{d^2F}{d\eta_1 d\phi} + P \frac{dF}{d\phi} = 0 \quad \ldots\ldots\ldots (21).$$

Differentiate this with respect to ψ, and notice that $\frac{d^2F}{d\phi\, d\psi}$ vanishes; we shall obtain

$$\frac{dP}{d\psi} \cdot \frac{dF}{d\phi} = 0,$$

of which the first factor alone can vanish. Thus P does not contain ψ; *i.e.*, P does not contain ζ.*

The integration of (21) with respect to η_1 and ϕ in succession gives

$$\zeta \equiv F = e^{-\int P d\eta_1} \cdot \phi(\xi_1, \ldots \xi_{n-1}) + \text{parts not involving } \phi.$$

13. The coefficient of $\left(\frac{d\psi}{d\eta_1}\right)$ gives the condition

$$\Sigma A_i \frac{d^2F}{d\xi_i d\psi} + Q \frac{dF}{d\psi} = 0 \quad \ldots\ldots\ldots (22).$$

Since $A_i \ldots$ do not involve ζ we can show by differentiating this equation with respect to ϕ, that $\frac{dQ}{d\phi}$ vanishes, or that Q does not contain ζ.

Also, from (22), we have $\frac{dF}{d\psi} = H$,

$$F = H\psi(\eta_1, \eta_2, \ldots \eta_{n-1}) + \text{a part not involving } \psi,$$

---

* This is wrong. Suppose the solution involves a function ψ_1 defined by the equation $\frac{d}{d\eta_1}\psi_1 = \psi$. Then in $\frac{d\zeta}{d\eta_1}$ we have a term $\frac{dF}{d\psi_1}\psi$; and in $\frac{d^2\zeta}{d\xi_i d\eta_1}$ a term $\frac{d^2F}{d\psi_1 d\phi}\psi\frac{d\phi}{d\xi_i}$. Thus the first equation of Art. 12 should be

$$A_i \left\{ \frac{d^2F}{d\eta_1 d\phi} + \frac{d^2F}{d\psi_1 d\phi}\psi + \ldots \right\} + P_i \frac{dF}{d\phi} = 0.$$

This does not affect the first result of the article; but the second result is true if, and only if, $\frac{d^2F}{d\psi_1 d\phi}$ vanishes. A corresponding correction must be made in Art. 13.

(*July*, 1876.)

where H is a solution of the equation

$$\Sigma A_i \frac{dH}{d\xi_i} + QH = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (23).$$

14. The coefficient of $\frac{d\psi}{d\eta_j}$ gives

$$\Sigma_i A_i \left\{ \frac{d^2\eta_j}{d\xi_i\, d\eta_i} + P \frac{d\eta_j}{d\xi_i} \right\} = 0.$$

The second set of terms vanishes in virtue of (15), of which η_j is a solution. The remaining condition

$$\Sigma_i A_i \frac{d^2\eta_j}{d\xi_i\, d\eta_1} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (24)$$

may also be simplified by the use of (15); viz.,

$$\Sigma_i A_i \frac{d\eta_j}{d\xi_i} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (15).$$

From this we get $\Sigma_i A_i \frac{d^2\eta_j}{d\eta_1\, d\xi_i} + \Sigma \frac{dA_i}{d\eta_1} \cdot \frac{d\eta_j}{d\xi_i} = 0,$

so that (24) reduces to $\Sigma_i \frac{dA_i}{d\eta_1} \cdot \frac{d\eta_j}{d\xi_i} = 0.$

Now this equation has $n-2$ integrals ($\eta_2, \ldots \eta_{n-1}$) in common with (15), and it has the same $n-1$ independent variables. Hence it must be identical with (15), or $\mu A_i = \frac{dA_i}{d\eta_1}$;

whence the quantities A_i either are, or can, by moving a factor common to all, be made independent of η_1. The same is therefore true of $\eta_2, \ldots \eta_{n-1}$, the integrals of (15).

15. We have now to examine the form of Z′ in (20). It will be convenient to consider first the case in which the solution contains the arbitrary functions ϕ, ψ unaccompanied by any of their indefinite integrals. In this case the complete solution is from Arts. 12, 13,

$$\zeta = \zeta_0 + e^{-\int P d\eta_1} \cdot \phi(\xi_1, \ldots \xi_{n-1}) + H\psi(\eta_1, \ldots \eta_{n-1}) \ldots\ldots\ldots (25).$$

Here ζ, and therefore its derivatives, are linear with respect to ϕ, ψ, and A_i, P, Q do not involve ϕ, ψ. Hence Z′ must also be linear with respect to ϕ, ψ; that is, it is a linear function of ζ.

Accordingly (20) is of the form

$$\Sigma_i A_i \left(\frac{d^2\zeta}{d\xi_i\, d\eta_1} + P \frac{d\zeta}{d\xi_i} \right) + Q \frac{d\zeta}{d\eta_1} + Z\zeta + W = 0 \ldots\ldots\ldots (26),$$

where A_i, P, Q, Z, W do not involve ζ.

Now let us express that the coefficient of ϕ in (26) is zero. We

obtain the condition $$\Sigma_i A_i \frac{dP}{d\xi_i} = Z - PQ \quad \ldots\ldots\ldots\ldots\ldots\ldots (27).$$

Again, from the coefficient of ψ we get, after an easy reduction by means of (23), the equation $$\frac{dQ}{d\eta_1} = Z - PQ \quad \ldots\ldots\ldots\ldots\ldots\ldots (28).$$

Lastly, taking the part independent of arbitrary functions, we find that ζ_0 is any particular integral of (26).

The conditions (27), (28) may be obtained in two other ways, which put in evidence very clearly the sufficiency and necessity of these relations.

In the first place they are the conditions that the coefficients of the arbitrary functions in (25) satisfy (26) when $W = 0$. Thus they are necessary, since we may put either ϕ or ψ equal to 1, and the other equal to zero.

Secondly, they express the conditions that we can find a new dependent variable ζ' such that the transformed equation is

$$\Sigma A_i \frac{d}{d\xi_i} \cdot \frac{d\zeta'}{d\eta_1} = 0,$$

the solution of which is

$$\zeta' = \phi(\xi_1, \ldots \xi_{n-1}) + \psi(\eta_1, \ldots \eta_{n-1}).$$

Hence the sufficiency.

[The last mode of viewing the question gives also the meaning of these conditions separately in a convenient form. We can always transform (26) so that the P or Q or Z of the transformed equation may vanish (viz., by taking $\zeta e^{\int P d\eta_1}$, $\frac{\zeta}{H}$ or $\frac{\zeta}{\zeta_0}$ as new dependent variable). Then (27) expresses that in the transformed equation P, Z simultaneously vanish; (28) that Q, Z simultaneously vanish; and

$$\Sigma_i A_i \frac{dP}{d\xi_i} = \frac{dQ}{d\eta_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots (29)$$

expresses the same for the P, Q of the new equation.]

16. We now proceed to discuss (20) when its solution involves the integrals of ϕ, ψ. In this case z' need not be linear in ζ; for instance, it may be an exponential function of ζ. In the present paper, however, we shall limit the discussion to that of (26) when the conditions (27) (28) are not satisfied.

If the condition (27) be not satisfied, the coefficient of ϕ in (26) does not vanish; the complete value of ζ must therefore involve some quantity such that the differentiations implied in (26) will give rise to a new term of the form $k \cdot \phi$. But the operator $\frac{d}{d\eta_1}$ does not affect ϕ. Thus,

the new part of ζ must involve some function of ϕ_1 where ϕ_1 is defined by the equation

$$\Sigma_i A_i \frac{d}{d\xi_i}.\ \phi_1 = \phi.$$

It is easily shown that ϕ_1 occurs in the first power in ζ. For suppose we had $\zeta = X_0\phi + X_1\phi_1 + X^1 f(\phi_1) + \ldots$

where $X_0 \equiv e^{-\int P d\eta_1}$ and $f(\phi_1)$ is not linear in ϕ_1. The coefficient of ϕ in (26) corresponding to this amended value of ζ consists of two parts, one of which involves the arbitrary factor $f'(\phi_1)$. Each of these must vanish separately, and we get

$$\frac{dX'}{d\eta_1} + PX' = 0,$$

whence $$X' = e^{-\int P d\eta_1} = X_0,$$

and $$\frac{dX_1}{d\eta_1} + PX_1 + \Sigma_i A_i \left(\frac{d^2 X_0}{d\xi_i d\eta_1} + P\frac{dX_0}{d\xi_i}\right) + Q\frac{dX_0}{d\eta_1} + ZX_0 = 0.$$

The first of these equations shows that any non-linear part of the new terms in ζ may be merged in ϕ. The second gives an equation for the determination of X_1 the coefficient of ϕ_1 in ζ.

If now the coefficient of ϕ_1 in (26) vanishes, the equation is complete: if not, we must add a term involving ϕ_2, where

$$\Sigma A_i \frac{d}{d\xi_i}\phi_2 \equiv \phi_1 \quad \text{or} \quad \left(\Sigma A_i \frac{d}{d\xi_i}\right)^2 \phi_2 \equiv \phi.$$

This may be shown to be linear as above; and the coefficient is calculated in the same way.

Precisely the same reasoning leads to similar results for the series commencing with $H\psi$. We may write the complete solution, therefore, in the form

$$\zeta = \zeta_0 + \sum_{j=0}^{j=k} X_j \phi_j + \sum_{j=0}^{j=l} H_j \psi_j \quad \ldots\ldots\ldots\ldots\ldots\ldots (30),$$

where $$\left(\Sigma A_i \frac{d}{d\xi_i}\right)^j \phi_j \equiv \phi_0,$$

$$\left(\frac{d}{d\eta_1}\right)^j \psi_j \equiv \psi_0,$$

$$\frac{dX_0}{d\eta} + PX_0 = 0, \quad \Sigma A_i \frac{dH_0}{d\xi_i} + QH_0 = 0,$$

$$\frac{dX_{j+1}}{d\zeta_1} + PX_{j+1} + \overline{X}_j = 0,$$

$$\Sigma_i A_i \frac{dH_{j+1}}{d\xi_i} + QH_{j+1} + \overline{H}_j = 0,$$

where we have written $\overline{X}_j$, $\overline{H}_j$ for the result of substituting X_j, H_j in (26) when $W=0$.

Lastly, we must have X_k, H_e satisfying the equation (26) when $W=0$.

As it is unsatisfactory to employ inverse operators in the solution, we will point out that these may at once be removed if the solution be finite. For if we replace ϕ_k and ψ_e by new arbitrary functions Φ, Ψ, (30) may be written in the form

$$\zeta = \zeta_0 + \sum_{j=0}^{j=k} X_j \left(\Sigma_i A_i \frac{d}{d\xi_i}\right)^{k-j} \Phi\ (\xi_1, \dots \xi_{n-1}) + \sum_{j=0}^{j=l} H_j \left(\frac{d}{d\eta_1}\right)^{l-j} (\Psi\eta_1, \eta_2 \dots \eta_{n-1}) \quad \dots\dots\dots\dots (31),$$

a solution in which only differentiations of the arbitrary functions occur.

17. *Example.* $\dfrac{d^2z}{dx_1 dx_2} + \dfrac{d^2z}{dx_1 dx_3} - \dfrac{4z}{(x_1+x_2+x_3)^2} = 0.$

Using the notation of Art. 16, we have

$$P = 0, \quad Q = 0,$$
$$X_0 = 1, \quad H_0 = 1,$$
$$\overline{X}_0 = \overline{H}_0 = \frac{-4}{(x_1+x_2+x_3)^2}$$

so that neither series of integrals is complete in one term.

For X_1 we have

$$\frac{dX_1}{dx_1} - \frac{4}{(x_1+x_2+x_3)^2} = 0,$$
$$X_1 = -\frac{4}{x_1+x_2+x_3},$$
$$\text{and } \overline{X}_1 = 0,$$

so that the solution is complete as far as one series of integrals is concerned.

For H_1 the equation

$$\frac{dH_1}{dx_2} + \frac{dH_1}{dx_2} - \frac{4}{(x_1+x_2+x_2)^2} = 0$$

gives $$H_1 = -\frac{2}{x_1+x_2+x_3},$$

whence $$\overline{H}_1 = 0,$$

or H_1 satisfies the original equation.

The solution is therefore

$$z = \phi\ (x_2, x_3) - \frac{4}{x_1+x_2+x_3}\left(\frac{d}{dx_2}+\frac{d}{dx_3}\right)^{-1}\phi\ (x_2, x_3) + \psi\ (x_1, x_2-x_3) - \frac{2}{x_1+x_2+x_3}\left(\frac{d}{dx_1}\right)^{-1}\psi\ (x_1, x_2-x_3);$$

or changing the arbitrary functions, and arranging,

$$z = \frac{2}{x_1+x_2+x_3}\{\phi(x_2, x_3) + \psi(x_1, x_2-x_3)\} - \frac{1}{2}\cdot\frac{d\phi}{dx_1} - \frac{1}{2}\frac{d\phi}{dx_2} - \frac{d\psi}{dx_1}.$$

18. In conclusion, I may be permitted to notice that a similar method is applicable without difficulty, though not without a considerable amount of labour in the general case, to equations of an order higher than the second. An equation of the m^{th} order, and having n independent variables, when it is completely soluble in terms of arbitrary functions of definite functions of the variables, will involve m arbitrary functions each having $n-1$ arguments. If the arguments are the same in all the functions, the given equation is transformable into an ordinary equation of the m^{th} order.

If n of the arguments are independent, and none of the integrals appear in the solution, it appears likely that the equation can be expressed in the form

$$\Delta_1 . \Delta_2 \ldots \Delta_m \zeta = 0,$$

where

$$\Delta \equiv A\frac{d}{dx_1} + B\frac{d}{dx_2} + \ldots + N\frac{d}{dx_n}$$

and where Δ_1, Δ_2, &c., are commutative. That this is sufficient is clear from the fact that the solution can be found, viz., it is

$$\zeta = \sum_{k=1}^{k=m} \phi_k,$$

where ϕ_k is an arbitrary function whose arguments are the integrals of

$$\Delta_k u = 0.$$

The necessity of this condition, however, is not obvious.

If i of the arguments should be common to all the functions ϕ_k, the given equation is reducible to an equation of the m^{th} order having $m-i$ independent variables.

January 13th, 1876.

LORD RAYLEIGH, F.R.S., Vice-President, in the Chair.

Major J. R. Campbell, Mr. R. Forsyth Scott, and Professor H. W. Lloyd Tanner, were admitted into the Society.

The following communications were made to the Society:—Mr. J. W. L. Glaisher, "On an Elliptic Function Identity;"—Professor Tanner, on "The Solution of Partial Differential Equations of the second order with any number of variables, when there is a complete

first integral;"—Professor Clifford on "Free Motion of a rigid system in an n-fold homaloid; expression of the velocities by Abelian functions;"—Lord Rayleigh on "The approximate solution of certain Potential Problems."

The following presents were received:—

"Une réforme géométrique: introduction à la géométrie descriptive des cristalloïdes, par le comte Léopold Hugo." Paris, 1874.

"Géométrie hugodomoidale anhellénique, mais philosophique et architectonique: la question de l'équidomoïde et des cristalloïdes géométriques, par le comte Léopold Hugo," Paris, 1875: from the Author.

"Ueber die v. Staudt'schen Würfe, von Rud. Sturm in Darmstadt" (Math. Annalen, Oct. 1875).

"Bulletin de la Société Mathématique de France." Tome iii., Juin, No. 2. Paris, 1875.

"Proceedings of Royal Society." Vol. xxiv., Nos. 164, 165.

"Bulletin des Sciences Mathématiques et Astronomiques." Tome ix., Sept., Oct., 1875.

"Annali di Matematica." Serie ii^a. Tomo vii° Fasc. 2°. (Nov. 1875, Mi ano.)

"Ueber eine Relation zwischen den Singularitäten einer algebraischen Curve, von Felix Klein in München" (aus den Sitzungsberichten der Physik.-Med.-Societät zu Erlangen, 13 December, 1875).

"Ueber den Zusammenhang der Flächen, von F. Klein in München." From the author.

"The Theory of Screws; a Study in the Dynamics of a Rigid Body." By Dr. R. Stawell Ball. 1876. From the author.

"Monatsbericht," Sept., Oct., Nov., 1875.

"Crelle." 81 Band, zweites Heft, 1875.

On an Elliptic-Function Identity. By J. W. L. Glaisher, M.A., F.R.S.

[*Read January 13th*, 1876.]

1. The identity in question is that

$$e^{4\left\{\frac{q}{1-q}\cos x+\frac{1}{3}\frac{q^3}{1-q^6}\cos 3x+\frac{1}{5}\frac{q^5}{1-q^{10}}\cos 5x+\&c.\right\}}$$
$$=\frac{1+\dfrac{4q}{1+q^2}\cos x+\dfrac{4q^2}{1+q^4}\cos 2x+\dfrac{4q^3}{1+q^6}\cos 3x+\&c.}{1-\dfrac{4q^2}{1+q^4}+\dfrac{4q^4}{1+q^8}-\dfrac{4q^6}{1+q^{12}}+\&c.} \quad \ldots\ldots\ldots(1);$$

and it is noteworthy on account of the peculiarity, that although the exponent on the left-hand side changes sign if q be replaced by q^{-1}, yet

the right-hand side remains unaltered. It is easily seen that for the truth of the identity q must be less than unity.

2. The identity (1) is obtained at once by equating the values of $\Delta\,\mathrm{am}\,\frac{2\mathrm{K}x}{\pi}$ given by the equations

$$\log \Delta\,\mathrm{am}\,\frac{2\mathrm{K}x}{\pi} =$$

$$= \log\sqrt{k'} + \frac{4q}{1-q^2}\cos 2x + \frac{4q^3}{3\,(1-} \cos 6x + \frac{4q^5}{5\,(1-q^{10})}\cos 10x + \&c. \quad \ldots\ldots\ldots(2);$$

$$\Delta\,\mathrm{am}\,\frac{2\mathrm{K}x}{\pi} =$$

$$= \frac{\pi}{2\mathrm{K}}\left\{1 + \frac{4q}{1+q^2}\cos 2x + \frac{4q^2}{1+q^4}\cos 4x + \frac{4q^3}{1+q^6}\cos 6x + \&c.\right\} \ldots(3);$$

replacing $\frac{2\sqrt{k'}\,\mathrm{K}}{\pi}$ by

$$1 - \frac{4q^2}{1+q^4} + \frac{4q^4}{1+q^8} - \frac{4q^6}{1+q^{12}} + \&c.,$$

and finally writing $\frac{1}{2}x$ for x.

3. It is easy to satisfy oneself that for the truth of (1) it is necessary that q should be less than unity; and if we examine the method by which (2) is obtained, it will appear that q is in the process assumed to be < 1, and that if $q > 1$ then the series would have the contrary sign. For the value of $\Delta\,\mathrm{am}\,\frac{2\mathrm{K}x}{\pi}$ with which we originally start is

$$\Delta\,\mathrm{am}\,\frac{2\mathrm{K}x}{\pi} =$$

$$= \sqrt{k'}\,\frac{(1+2q\cos 2x+q^2)(1+2q^3\cos 2x+q^6)(1+2q^5\cos 2x+q^{10})\ldots}{(1-2q\cos 2x+q^2)(1-2q^3\cos 2x+q^6)(1-2q^5\cos 2x+q^{10})\ldots}$$

giving rise to the identity

$$\frac{(1+2q\cos 2x+q^2)(1+2q^3\cos 2x+q^6)(1+2q^5\cos 2x+q^{10})\ldots}{(1-2q\cos 2x+q^2)(1-2q^3\cos 2x+q^6)(1-2q^5\cos 2x+q^{10})\ldots}$$

$$= \frac{1 + \frac{4q}{1+q^2}\cos 2x + \frac{4q^2}{1+q^4}\cos 4x + \frac{4q^3}{1+q^6}\cos 6x + \&c.}{1 - \frac{4q^2}{1+q^4} + \frac{4q^4}{1+q^8} - \frac{4q^6}{1+q^{12}} + \&c.} \quad \ldots\ldots(4),$$

both sides of which remain unaltered if q^{-1} be substituted for q, and which is true for all values of q.

Taking the logarithm of the left-hand side of (4), and if $q < 1$ expanding by the formulæ

$$\log\frac{1+2q\cos 2x+q^2}{1-2q\cos 2x+q^2}=4\left\{q\cos 2x+\frac{q^3}{3}\cos 6x+\&c.\right\}\ldots\ldots(5),$$

$$\log\frac{1+2q^3\cos 2x+q^6}{1-2q^3\cos 2x+q^6}=4\left\{q^3\cos 2x+\frac{q^9}{3}\cos 6x+\&c.\right\}\ldots\ldots(6),$$

... ...

we have by addition, P denoting the left-hand side of (4),

$$\log \mathrm{P}=\frac{4q}{1-q^2}\cos 2x+\frac{4q^3}{3(1-q^6)}\cos 6x+\&c.\ldots\ldots\ldots\ldots(7);$$

and although the series in (5), (6), ... are only convergent if $q<1$, yet the series for log P, obtained by simple summation, is convergent, whatever q may be.

But if q be >1, the process would be

$$\log\frac{1+2q\cos 2x+q^2}{1-2q\cos 2x+q^2}=\log\frac{1+2q^{-1}\cos 2x+q^{-2}}{1-2q^{-1}\cos 2x+q^{-2}}$$

$$=4\left\{q^{-1}\cos 2x+\frac{q^{-3}}{3}\cos 6x+\&c.\right\},$$

... ...

so that $\log \mathrm{P}=\dfrac{4q^{-1}}{1-q^{-2}}\cos 2x+\dfrac{4q^{-3}}{3(1-q^{-6})}\cos 6x+\&c.$

$$=-\left\{\frac{4q}{1-q^2}\cos 2x+\frac{4q^3}{3(1-q^6)}\cos 6x+\&c.\right\},$$

which completely accounts for the change of sign produced in the series which forms the exponent of e in (1) when q is replaced by q^{-1}. In fact, although the equations (5), (6) ... are obviously untrue if $q>1$, as they are then divergent, yet the equation (7) derived from them, *though no less untrue, is not obviously so, as it is always convergent.*

4. I here add an algebraical proof of the identity (4), which is similar in character to that given by Jacobi ("Fundamenta Nova," p. 180) of the expansion of

$$(1-2q\cos 2x+q^2)(1-2q^3\cos 2x+q^6)(1-2q^5\cos 2x+q^{10})\ldots$$

in a series of cosines of multiples of x.

It is necessary first to prove the two following lemmas:

I.
$$\frac{1+qz\,.\,1+q^3z\,.\,1+q^5z\ldots}{1-qz\,.\,1-q^3z\,.\,1-q^5z\ldots}$$

$$=1+\frac{2q}{1-q^2}z+\frac{2q^2\,.\,1+q^2}{1-q^2\,.\,1-q^4}z^2+\frac{2q^3\,.\,1+q^2\,.\,1+q^4}{1-q^2\,.\,1-q^4\,.\,1-q^6}z^3+\&c.$$

II. $$\frac{1+q\,.\,1+q^2\,.\,1+q^3\,\ldots}{1-q\,.\,1-q^2\,.\,1-q^3\,\ldots}\;\frac{1+qz\,.\,1+q^2z\,.\,1+q^3z\,\ldots}{1-qz\,.\,1-q^2z\,.\,1-q^3z\,\ldots}$$

$$=1+\frac{2q}{1-q}\,\frac{1+z}{1-qz}+\frac{2q^2\,.\,1+q}{1-q\,.\,1-q^2}\,\frac{1+z\,.\,1+qz}{1-qz\,.\,1-q^2z}$$

$$+\frac{2q^3\,.\,1+q\,.\,1+q^2}{1-q\,.\,1-q^2\,.\,1-q^3}\,\frac{1+z\,.\,1+qz\,.\,1+q^2z}{1-qz\,.\,1-q^2z\,.\,1-q^3z}+\&c.$$

Lemma I. is easily obtained by assuming

$$\frac{1+qz\,.\,1+q^3z\,\ldots}{1-qz\,.\,1-q^3z\,\ldots}=1+A_1z+A_2z^2+A_3z^3+\&c.,$$

putting q^2z for z, and equating coefficients in the equation

$$(1+qz)(1+A_1q^2z+A_2q^4z^2+\&c.)=(1-qz)(1+A_1z+A_2z^2+\&c.).$$

The truth of lemma II. is best seen by remarking that

$$\frac{1+z}{1-qz}=\frac{q-1}{2q}+\frac{q+1}{2q}\,.\,\frac{1+qz}{1-qz},$$

$$\frac{1+z}{1-q^2z}=\frac{q^2-1}{2q^2}+\frac{q^2+1}{2q^2}\,.\,\frac{1+q^2z}{1-q^2z};$$

$$\ldots\qquad\ldots\qquad\ldots\qquad\ldots$$

whence $$1+\frac{2q}{1-q}\,.\,\frac{1+z}{1-qz}=1+\frac{2q}{1-q}\,.\,\frac{q-1}{2q}+\frac{2q}{1-q}\,.\,\frac{q+1}{2q}\,.\,\frac{1+qz}{1-qz}$$

$$=\frac{1+q}{1-q}\,.\,\frac{1+qz}{1-qz},$$

$$1+\frac{2q}{1-q}\,.\,\frac{1+z}{1-qz}+\frac{2q^2\,.\,1+q}{1-q\,.\,1-q^2}\,.\,\frac{1+z\,.\,1+qz}{1-qz\,.\,1-q^2z}$$

$$=1+\frac{2q}{1-q}\,.\,\frac{q-1}{2q}+\frac{2q}{1-q}\left\{\frac{q+1}{2q}+\frac{q\,.\,1+q}{1-q^2}\,.\,\frac{q^2-1}{2q^2}\right\}\frac{1+qz}{1-qz}$$

$$+\frac{2q^2\,.\,1+q}{1-q\,.\,1-q^2}\,.\,\frac{q^2+1}{2q^2}\,.\,\frac{1+qz\,.\,1+q^2z}{1-qz\,.\,1-q^2z}$$

$$=\frac{1+q\,.\,1+q^2}{1-q\,.\,1-q^2}\,.\,\frac{1+qz\,.\,1+q^2z}{1-qz\,.\,1-q^2z},$$

and so on, the constant term and the coefficients of

$$\frac{1+qz}{1-qz},\quad\frac{1+qz\,.\,1+q^2z}{1-qz\,.\,1-q^2z},\quad\ldots$$

all vanishing except the last.

5. The coefficient of z^n (which is, of course, equal to that of z^{-n}) in the product

$$\frac{1+qz\,.\,1+q^3z\,.\,1+q^5z\,\ldots}{1-qz\,.\,1-q^3z\,.\,1-q^5z\,\ldots}\;\frac{1+qz^{-1}\,.\,1+q^3z^{-1}\,.\,1+q^5z^{-1}\,\ldots}{1-qz^{-1}\,.\,1-q^3z^{-1}\,.\,1-q^5z^{-1}\,\ldots}$$

is, by lemma I., equal to the coefficient of z^n in

$$\left\{1+\frac{2q}{1-q^2}z+\frac{2q^2.1+q^2}{1-q^2.1-q^4}z^2+\frac{2q^3.1+q^2.1+q^4}{1-q^2.1-q^4.1-q^6}z^3+\&c.\right\}$$
$$\times\left\{1+\frac{2q}{1-q^2}z^{-1}+\frac{2q^2.1+q^2}{1-q^2.1-q^4}z^{-2}+\frac{2q^3.1+q^2.1+q^4}{1-q^2.1-q^4.1-q^6}z^{-3}+\&c.\right\},$$

and is therefore

$$=\frac{2q^n.1+q^2.1+q^4\ldots1+q^{2n-2}}{1-q^2.1-q^4\ldots1-q^{2n}}$$
$$\times\left\{1+\frac{2q^2}{1-q^2}.\frac{1+q^{2n}}{1-q^{2n+2}}+\frac{2q^4.1+q^2}{1-q^2.1-q^4}.\frac{1+q^{2n}.1+q^{2n+2}}{1-q^{2n+2}.1-q^{2n+4}}+\&c.\right\}$$
$$\ldots\ldots\ldots(8);$$

and by lemma II. (putting therein q^2 for q, and taking $z=q^{2n}$), the expression in brackets

$$=\frac{1+q^2.1+q^4.1+q^6\ldots}{1-q^2.1-q^4.1-q^6\ldots}\,\frac{1+q^{2n+2}.1+q^{2n+4}.1+q^{2n+6}\ldots}{1-q^{2n+2}.1-q^{2n+4}.1-q^{2n+6}\ldots},$$

so that the coefficient of z^n

$$=\frac{2q^n}{1+q^{2n}}\left(\frac{1+q^2.1+q^4.1+q^6\ldots}{1-q^2.1-q^4.1-q^6\ldots}\right)^2,$$

and we have identically

$$\frac{1+qz.1+q^3z.1+q^5z\ldots}{1-qz.1-q^3z.1-q^5z\ldots}\,\frac{1+qz^{-1}.1+q^3z^{-1}.1+q^5z^{-1}\ldots}{1-qz^{-1}.1-q^3z^{-1}.1-q^5z^{-1}\ldots}$$
$$=\left\{\frac{1+q^2.1+q^4.1+q^6\ldots}{1-q^2.1-q^4.1-q^6\ldots}\right\}^2$$
$$\times\left\{1+\frac{2q}{1+q^2}(z+z^{-1})+\frac{2q^2}{1+q^4}(z^2+z^{-2})+\&c.\right\},$$

whence putting $z=e^{2ix}$, and denoting the former of the two factors last written by Q^2,

$$\frac{(1+2q\cos2x+q^2)(1+2q^3\cos2x+q^6)\ldots}{(1-2q\cos2x+q^2)(1-2q^3\cos2x+q^6)\ldots}$$
$$=Q^2\left\{1+\frac{4q}{1+q^2}\cos2x+\frac{4q^2}{1+q^4}\cos4x+\&c.\right\}.$$

Put $x=\frac{1}{4}\pi$, and we find

$$\frac{1}{Q^2}=1-\frac{4q^2}{1+q^4}+\frac{4q^4}{1+q^8}-\&c.,$$

thus proving (4); and as the logarithm of the left-hand side of (4) has been obtained in § 3, we thus have a complete algebraical demonstration of the original identity (1).

6. In regard to lemma II., which was of course suggested by the form of the quantity within the brackets in (8), it is worth noticing

the manner in which the coefficients are determined by the ordinary process of replacing z by qz. Assuming

$$\frac{1+q\,.\,1+q^2\,\ldots}{1-q\,.\,1-q^2\,\ldots}\,\frac{1+qz\,.\,1+q^2z\,\ldots}{1-qz\,.\,1-q^2z\,\ldots}$$
$$= 1+\mathrm{A}_1\frac{1+z}{1-qz}+\mathrm{A}_2\frac{1+z\,.\,1+qz}{1-qz\,.\,1-q^2z}+\&c.,$$

and using, as in § 5,

$$\frac{1+z}{1-qz}=\frac{q-1}{2q}+\frac{q+1}{2q}\,\frac{1+qz}{1-qz},\ \&c.,$$

(the object being to transform the terms involving z, so that when qz is written for z, and the series is multiplied by $\frac{1+qz}{1-qz}$, they shall remain unaltered in form), we find as the equation from which to determine A_1, A_2 ...

$$\left[1+\frac{q-1}{2q}\mathrm{A}_1\right]+\left[\frac{q+1}{2q}\mathrm{A}_1+\frac{q^2-1}{2q^2}\mathrm{A}_2\right]\frac{1+qz}{1-qz}$$
$$+\left[\frac{q^2+1}{2q^2}\mathrm{A}_2+\frac{q^3-1}{2q^3}\mathrm{A}_3\right]\frac{1+qz\,.\,1+q^2z}{1-qz\,.\,1-q^2z}+\&c.$$
$$=\frac{1+qz}{1-qz}\left\{\left[1+\frac{q-1}{2q}\mathrm{A}_1\right]+\left[\frac{q+1}{2q}\mathrm{A}_1+\frac{q^2-1}{2q^2}\mathrm{A}_2\right]\frac{1+q^2z}{1-q^2z}+\&c.\right\};$$

whence the quantities within square brackets must vanish, thus giving the true values of A_1, A_2 ...; and this is, in fact, the manner in which I first obtained them. But the result itself—viz., that if the series be expressed in the form

$$\mathrm{B}_0+\mathrm{B}_1\frac{1+qz}{1-qz}+\mathrm{B}_2\frac{1+qz\,.\,1+q^2z}{1-qz\,.\,1-q^2z}+\&c.\ \ldots\ldots\ldots\ldots\ldots(9),$$

then the values of B_0, B_1 ... are zero, so that the series apparently vanishes identically—seems for the moment curious. The explanation has been, in fact, already given in § 4, where it is shown that the *last* (the $n+1^{\text{th}}$) term in square brackets is $\frac{q^n+1}{2q^n}\mathrm{A}_n$, and not $\frac{q^n+1}{2q^n}\mathrm{A}_n+\frac{q^{n+1}-1}{2q^{n+1}}\mathrm{A}_{n+1}$; or otherwise, in the expansion of

$$\mathrm{A}\,\frac{1+qz\,.\,1+q^2z\,\ldots\,1+q^nz}{1-qz\,.\,1-q^2z\,\ldots\,1-q^nz}$$

in the form (9), it is clear that B_0, B_1 ... all vanish except B_n, which $=\mathrm{A}$, and this is true however great n may be.

On the Free Motion under no Forces of a Rigid System in an n-fold Homaloid. (Provisional Notice.) *By* Prof. W. K. Clifford, M.A., F.R.S.

[*Read January* 13*th*, 1876.]

The problem of the rotation under no forces of a rigid body about a fixed point in ordinary three-dimensional space is the same as the problem of free motion under no forces; for the motion about the centre of inertia takes place as if it were a fixed point. But it is also the same thing as the problem of the free motion of a rigid system on the surface of a sphere, or in elliptic space of two dimensions.* So also the problem of the free motion of a solid in elliptic space of three dimensions is the same as that of the free motion, or motion about a fixed point, in parabolic or homaloidal space of four dimensions. And, in general, the problem of free motion in elliptic space of n dimensions is identical with that of free motion, or motion about a fixed point, in parabolic space of $n+1$ dimensions.

The form of the problem which is considered in what follows is that which deals with the motion about a fixed point in parabolic space of n dimensions.

I.

Let the coordinates of a point, referred to a rectangular system, be $x_1, x_2, \ldots x_n$. If this point belongs to a rigid system in motion, its velocity is given by the equations

$$\dot{x}_h = \Sigma p_{hk} x_k \quad (h, k = 1, 2, \ldots n) \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where $p_{hk} = -p_{kh}$, $p_{hh} = 0$. The $\frac{1}{2}n(n-1)$ quantities p are of the nature of rotational velocities of the rigid system. It may be observed that if the vector from the origin to the point x be represented in terms of n unit vectors $\iota_1 \iota_2 \ldots \iota_n$, satisfying the equations $\iota_h \iota_k = -\iota_k \iota_h$, $\iota_h^2 = -1$, then the velocity of the rigid body may be represented in terms of the $\frac{1}{2}n(n-1)$ products $\iota_h \iota_k$; namely, we may write

$$\rho = \Sigma \iota_h x_h, \quad -2p = \Sigma p_{hk} \iota_h \iota_k,$$

and then the equation (1) may be put into the form

$$\dot{\rho} = \nabla p \rho.$$

If dm be the element of mass at the point x, its kinetic energy is

$$\tfrac{1}{2}\Sigma \dot{x}^2 dm = \tfrac{1}{2} dm \left[\Sigma p_{hk}^2 (x_h^2 + x_k^2) + 2\Sigma p_{hl} p_{kl} x_h x_k\right].$$

Let then $$\int \dot{x}_h^2 \, dm = \alpha_h, \quad \int x_h x_k \, dm = \beta_{hk},$$

* According to Dr. Klein's nomenclature, a space every point of which can be uniquely represented by a set of values of n variables is called elliptic, parabolic, or hyperbolic, when its curvature is uniform and positive, zero, or negative. The geometry of the sphere becomes elliptic when opposite points are regarded as identical.

the integrations extending over the whole rigid system; then, if T be the kinetic energy of the system,

$$2T = \Sigma (a_h + a_k) p_{hk}^2 + 2\Sigma \beta_{hk} p_{hl} p_{kl} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Write now $$q_{hk} = \frac{\delta T}{\delta p_{hk}}, \quad -2q = \Sigma q_{hk} \iota_h \iota_k,$$

then q is the *momentum* of the system; it is a linear function of the velocity, or $q = \phi(p)$, and twice the kinetic energy is the scalar part of the product of velocity and momentum, $2T = Spq$. The equations of motion are $\dot{q} = f$, where f is the system of applied forces, or in the present case of no forces $\dot{q} = 0$; viz., this is equivalent to $\frac{1}{2}n(n-1)$ equations. From this we get the first integrals, $q =$ constant, and (since $0 = Sp\dot{q} = 2\dot{T}$) $T =$ constant.

But these equations are inconvenient, because the α and β are variable, depending upon the position of the body. We must therefore follow Euler in referring the motion to axes moving with the body, and coinciding with the principal axes at the fixed point. This will make all the β vanish; and we shall have $q_{hk} = (a_h + a_k) p_{hk}$.

From equation (1) we obtain

$$\dot{q}_{hk} = \int dm (\ddot{x}_h x_k - x_h \ddot{x}_k) = \Sigma_l \int dm [\dot{x}_l (p_{hl} x_k - p_{kl} x_h) + x_l (\dot{p}_{hl} x_k - \dot{p}_{kl} x_h)].$$

Assuming that $\int dm\, x_h x_k = 0$ when h and k are different, and remembering that $\dot{q}_{hk} = 0$, we may write this equation in the form

$$(a_h + a_k) \dot{p}_{hk} + (a_h - a_k) \Sigma_l p_{hl} p_{lk} = 0 \quad \ldots\ldots\ldots\ldots\ldots (3).$$

Here $\dot{p}_{hk}$ means the rate of change of that component of velocity which coincides at the moment with one of the principal components; it must be distinguished from the rate of change of the principal component, which we shall call $(\dot{p})_{hk}$. In general, if the system of axes has the rotational velocities r_{hk}, we shall have

$$\dot{p}_{hk} = (\dot{p})_{hk} + \Sigma (p_{hl} r_{kl} - p_{kl} r_{hl});$$

but in the present case the r are equal to the p, because the axes move with the body, and so $\dot{p}_{hk} = (\dot{p})_{hk}$. Thus in the equations (3), which are analogous to Euler's equations for three dimensions, the symbols $\dot{p}_{hk}$ may be understood to mean the rates of change of the principal components of rotation.

If the symbol V_2 is regarded as selecting all the binary products of $\iota_1 \iota_2 \ldots \iota_n$ out of any rational integral function of them, the last equations may be written $\dot{q} = V_2 pq$;

and it may be observed that we have also

$$q = \int dm\, V_2 \rho \dot{\rho}.$$

II.

These equations may be integrated by means of Θ-functions of $n-2, = s$, arguments, one of which is a linear function of the time. It is most convenient to use these in the form employed by Göpel for the case $s=2$. Let $u_1, u_2, \ldots u_s$ be the arguments, and let $B_{11}, B_{12}, \ldots B_{1s}, \ldots B_{s1}, \ldots B_{ss}$ be s^2 constant quantities, and let

$$U_h = (u_h + 2m_1 B_{1h} + 2m_2 B_{2h} + \ldots + 2m_s B_{sh})^2 = (u_h + 2\Sigma m_k B_{kh})^2;$$

then we shall write

$$G(u_1, u_2, \ldots u_s) = \Sigma_m e^{\Sigma U_h},$$

where the whole numbers m are to take independently all values from $-\infty$ to $+\infty$. It is clear that the function G is unaltered if we simultaneously increase $u_1, u_2, \ldots u_s$ by equimultiples of the quantities $2B_{h1}, 2B_{h2}, \ldots 2B_{hs}$, for this is only increasing m_h by an integer. Moreover, if we determine s^2 quantities A so that

$$4\Sigma_k A_{hk} B_{hk} = \pi i, \quad \Sigma_k A_{hk} B_{lk} = 0,$$

then the function $e^{-\Sigma uu} G$ is unaltered if we simultaneously increase $u_1, u_2, \ldots u_s$ by equimultiples of $4A_{h1}, 4A_{h2}, \ldots 4A_{hs}$. In what follows we shall write for shortness $G(u+X_h)$ instead of $G(u_1+X_{h1}, u_2+X_{h2}, \ldots u_n+X_{hn})$, omitting always the last suffix, and mentioning only one argument u.

A linear function of the A, B with coefficients 0 or 1 will be called a *quadrant;* there are clearly 2^{2s} quadrants, if zero be included among them. Let X and Y be two quadrants, A_h the difference between those parts of them which involve the quantities A_{hk}, then the function

$$e^{-\Sigma A_u} \frac{G(u+X)}{G(u+Y)} = Al_{X,Y}(u)$$

is $2s$-periodic in the arguments u, the periods being 4A, 4B. It is convenient to speak of the *distance* of two quadrants X and Y, meaning the number of coefficients of the A and B which must be changed from 0 to 1, or *vice versâ* to make one of them into the other. This distance may be any of the numbers 1, 2, ... $2s$; and accordingly there are $2s$ really distinct $2s$-periodic functions.

It is, however, possible to form a group of $n, = s+2$, quadrants $X_1, X_2, \ldots X_n$ having such a relation to the quadrant O that if we write $Al_{hk}(u)$ for $e^{-\Sigma Au} G(u+X_h+X_k) : Gu$, the $\frac{1}{2}n(n-1)$ functions $Al_{hk}(u)$ satisfy the equations

$$\delta_u Al_{hk}(u) = \Sigma c_l Al_{hl}(u) \cdot Al_{kl}(u) \quad \ldots\ldots\ldots\ldots (4),$$

where the δ_u applies to any one of the arguments; but the values of the c will depend upon which argument is taken. In the case of the hyperelliptic functions, the four quadrants X may be taken to be the quantities A_1, A_2, B_1, B_2.

It appears therefore that the equations (3) may be integrated if we write $p_{hk} = \lambda_{hk} \mathrm{Al}_{hk}(u)$, where $u_1 = at + e$.

III.

From the $\frac{1}{2}n(n-1)$ equations (3) let us pick out $n-1$, namely

$$-(a_1+a_h)\dot{p}_{1h} = (a_1-a_h)\Sigma p_{hk} p_{1k};$$

if we write $(a_1+a_h)p_{1h} = (a_1-a_h)\xi_h$, these equations become

$$-\dot{\xi}_h = \Sigma p_{hk}\xi_k.$$

But these are the equations for the velocity of a fixed point ξ relative to the moving axes in $n-1$ dimensions. The rest of the equations (3), if we write in them $p_{1h} = 0$, become the equations for the component rotations in $n-1$ dimensions. Thus the solution for the rotational velocities *and* the position of a point fixed in space, for $n-1$ dimensions, are obtained by diminishing the number of periods in the solution for n dimensions; it consists accordingly of the Al functions expressing the rotation-velocities, and of Rosenhain's combinations of Θ-functions and exponentials, expressing the position.

On the Approximate Solution of certain Problems relating to the Potential. *By* Lord Rayleigh, M.A., F.R.S.

(*Read January* 13*th*, 1876.)

The first problem that I propose for consideration is that of the flow of electricity in two dimensions along a strip of uniform metal sheet, such as tinfoil, the strip being bounded by curves symmetrical with respect to a central line, taken as axis of x, and nowhere more than moderately inclined to that axis. The equations of the boundary may be written $y = \pm y_1$, where y_1 is a slowly varying function of x. On account of the symmetry the potential is an even function of y, and the axis of x a stream line, so that the problem is the same as if the axis were itself a boundary. The conditions to be satisfied by the potential ϕ are therefore the usual equation of Laplace,

$$\nabla^2\phi = \frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} = 0,$$

which must be true over the whole area of the strip, together with the boundary conditions, that there must be no normal flow across the curves $y = y_1$ and $y = 0$. If we introduce the stream

function ψ, which is related to ϕ by the equations

$$\frac{d\phi}{dx} = \frac{d\psi}{dy}, \qquad \frac{d\phi}{dy} = -\frac{d\psi}{dx} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and itself satisfies $\nabla^2\psi = 0$, the boundary conditions may be expressed by supposing ψ to be zero when $y = 0$, and when $y = y_1$ to assume a second constant value ψ_1.

If the boundary of the strip were straight and parallel to the axis, the current $\frac{d\phi}{dx}$ would be constant at all points, and we should have

$$\phi = f \,.\, x, \qquad \psi = f \,.\, y,$$

where f represents the constant current.

The object of the present investigation is to determine the values of ϕ and ψ, when the above simple values may be taken as a basis of approximation, and to calculate corrections for the effect of the deviation of y_1 from constancy.

Denoting by f the slowly varying magnitude of the current when $y = 0$, we may take

$$\left.\begin{aligned} \phi &= \cos\left(y\frac{d}{dx}\right) f_1 = f_1 - \frac{y^2}{1\,.\,2} f' + \frac{y^4}{1\,.\,2\,.\,3\,.\,4} f''' - \&c. \\ \psi &= \sin\left(y\frac{d}{dx}\right) f_1 = yf - \frac{y^3}{1\,.\,2\,.\,3} f'' + \&c. \end{aligned}\right\} \ldots (2),$$

where $f_1 = \int f\,dx$.

These values satisfy the general conditions of the potential and stream functions, and when $y = 0$ make

$$\frac{d\phi}{dx} = f, \qquad \psi = 0.$$

The second of equations (2) may be regarded as determining the lines of flow (any one of which may be supposed to be the boundary of the sheet) in terms of f. What remains to be effected is the determination of f from the condition that $\psi = \psi_1$, when $y = y_1$, viz.,

$$\psi_1 = y_1 f - \frac{y_1^3}{1\,.\,2\,.\,3} f'' + \frac{y_1^5}{1\,.\,2\,.\,3\,.\,4\,.\,5} f^{iv} - \&c.,$$

an equation which may be treated by the method of successive approximation. We have

$$f = \frac{\psi_1}{y_1} + \frac{y_1^2}{1\,.\,2\,.\,3} f'' - \frac{y_1^4}{1\,.\,2\,.\,3\,.\,4\,.\,5} f^{iv} + \&c. \ldots\ldots\ldots\ldots (3);$$

whence we obtain

$$\begin{aligned} f &= \frac{\psi_1}{y_1} + \frac{y_1^2}{6}\left[\frac{d^2}{dx^2}\left(\frac{\psi_1}{y}\right) + \frac{1}{6}\frac{d^2}{dx^2}\left\{y_1^2\frac{d^2}{dx^2}\left(\frac{\psi_1}{y_1}\right)\right\}\right] - \frac{y_1^4}{120}\frac{d^4}{dx^4}\left(\frac{\psi_1}{y_1}\right) \\ &= \frac{\psi_1}{y_1} + \frac{\psi_1}{6}\left(2y_1^{-1}y_1'^2 - y_1''\right) + \frac{y_1^2\psi}{36}\frac{d^2}{dx^2}\left(2y_1^{-1}y_1'^2 - y_1''\right) - \frac{y_1^4}{120}\frac{d^4}{dx^4}\left(\frac{\psi_1}{y_1}\right) \ldots (4), \end{aligned}$$

where accents denote differentiation with respect to x.

In order to calculate the electrical resistance of the strip, we must

compare the total current with the difference of potentials at the two extremities. If the lamina be of unit specific conductivity, the total current

$$= \int_0^{y_1} \frac{d\phi}{dx} dy = \int_0^{y_1} \frac{d\psi}{dy} dy = \psi_1.$$

For the potential on the axis we have

$$\phi = \int f\, dx,$$

and therefore the resistance of the strip is represented by

$$\frac{1}{\psi_1} \int f\, dx,$$

where f has the value given by (4.)

If the equipotential lines, between which the resistance is required, be situated in parts of the strip, where the edge is straight and parallel to the axis (for a sufficient distance in comparison with the width), our expression may be simplified by integration by parts, since all the differential coefficients of y_1 with respect to x may be supposed to vanish at the limits under the circumstances contemplated.

We have

$$\int y_1^2 \frac{d^2}{dx^2}\left(2y_1^{-1}y_1'^2 - y_1''\right) dx = \int \left(2y_1^{-1}y_1'^2 - y_1''\right) \frac{d^2y_1^2}{dx^2}\, dx$$

$$= 2\int \left(2y_1^{-1}y_1'^2 - y_1''\right)\left(y_1'^2 - y_1y_1''\right) dx,$$

$$\int y_1^4 \frac{d^4y_1^{-1}}{dx^4}\, dx = \int \frac{d^2y_1^4}{dx^2}\frac{d^2y_1^{-1}}{dx^2} dx$$

$$= 4\int \left(2y_1^{-1}y_1'^2 - y_1''\right)\left(3y_1'^2 + y_1y_1''\right) dx;$$

and the final result for the resistance may be written

$$\int \frac{dx}{y_1}\left\{1 + \frac{1}{3}y_1'^2 - \frac{1}{45}\left(2y_1'^2 - y_1y_1''\right)^2\right\} \quad \ldots\ldots\ldots\ldots\ldots(5).$$

This expression admits of various forms. For example, instead of the second term we might take

$$\int \frac{dx}{y_1}\left\{\ldots + \frac{1}{6}\left(2y_1^2 - y_1y_1''\right) - \ldots\right\},$$

which differs by the addition of a term which is a complete differential, and therefore disappears on integration.

The success of the approximation depends of course upon the degree of slowness with which y_1 varies, but it is not necessary to suppose that the whole variation of y_1 is small within the limits of x under consideration. If we suppose that y_1 is a function of ωx, where ω is a small quantity, our expression for the resistance will include all terms containing as factors lower powers of ω than ω^6.

The same expression (5) gives us the *conductivity* of the strip be-

tween the curves $y = y_1$ and $y = 0$, supposed equipotential. For it is a general proposition in the theory of conduction in two dimensions that if two opposite sides of any curvilinear quadrilateral be at uniform (through different) potentials and the other pair of sides be non-conducting, the resistance is the reciprocal of that corresponding to an altered state of things in which the first pair of opposite sides are non-conducting and the second pair are equipotential. Its truth is easily seen if we fill up the quadrilateral with the intermediate stream and equipotential curves, which will ultimately divide the area into elementary squares. The resistance of any square between opposite edges is independent of the size of the square, any augmentation of resistance due to an increased distance to be travelled being compensated by the effect of increased width. Thus, if the number of interpolated stream and equipotential curves be respectively $m-1$ and $n-1$, the whole resistance is to that of a square in the ratio $n : m$; while for the same reason, if the equipotential and stream lines be interchanged, the resistance is $m : n$ of that of a square.

The same result is applicable to other problems mathematically analogous; for example, the calculation of the capacity of an electrical accumulator.

A similar method applies to the investigation of conduction in three dimensions in a conductor, which is symmetrical round the axis of x, and whose boundary does not differ violently from a cylindrical surface. If r be the distance of any point from the axis, ϕ and ψ the potential and stream functions, we have

$$\frac{d\phi}{dx} = \frac{1}{r}\frac{d\psi}{dr}, \quad \frac{d\phi}{dr} = -\frac{1}{r}\frac{d\psi}{dx} \quad \text{.....(6),}$$

whence by elimination

$$\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} + \frac{d^2\phi}{dx^2} = 0 \quad \text{.....(7).}$$

$$\frac{d^2\psi}{dr^2} - \frac{1}{r}\frac{d\psi}{dr} + \frac{d^2\psi}{dx^2} = 0 \quad \text{.....(8),}$$

which shew that ϕ and ψ are not now, as in the case of two dimensions, interchangeable. The series corresponding to (2) are

$$\left.\begin{aligned} \phi &= \mathrm{F} - \frac{r^2\mathrm{F}''}{2^2} + \frac{r^4\mathrm{F}^{\mathrm{iv}}}{2^2.4^2} - \frac{r^6\mathrm{F}^{\mathrm{vi}}}{2^2.4^2.6^2} + \ldots\ldots \\ \psi &= \frac{r^2\mathrm{F}'}{2} - \frac{r^4\mathrm{F}'''}{2^2.4} + \frac{r^6\mathrm{F}^{\mathrm{v}}}{2^2.4^2.6} - \frac{r^8\mathrm{F}^{\mathrm{vii}}}{2^2.4^2.6^2.8} + \ldots \end{aligned}\right\} \quad \text{.....(9),}$$

where F is a function of x so far arbitrary. If the form of the boundary be defined by $r = y$, we have

$$\psi_1 = \frac{y^2\mathrm{F}'}{2} - \frac{y^4\mathrm{F}'''}{2^2.4} + \ldots\ldots,$$

from which F′ is to be determined in terms of y. By successive approximation we obtain

$$F' = \frac{2\psi_1}{y^2} + \frac{y^2}{8}\left\{\frac{d^2}{dx^2}\left(\frac{2\psi_1}{y^2}\right) + \frac{1}{8}\frac{d^2}{dx^2}y^2\frac{d^2}{dx^2}\left(\frac{2\psi_1}{y^2}\right)\right\} - \frac{y^4}{12\,.\,4^2}\frac{d^4}{dx^4}\left(\frac{2\psi_1}{y^2}\right) \quad \ldots\ldots\ldots\ldots(10).$$

The total stream is given by the integral

$$\int_0^y \frac{d\phi}{dx}\,2\pi r\,dr = \int_0^y \frac{1}{r}\frac{d\psi}{dr}\,.\,2\pi r\,dr = 2\pi\psi_1;$$

and therefore the resistance between any two equipotential surfaces is represented by

$$\frac{1}{2\pi\psi_1}\int F'dx.$$

On the supposition that at each limit the boundary is cylindrical for a sufficient distance in comparison with the diameter, we may treat this expression by integration by parts in the same manner as for the corresponding problem in two dimensions, and we then obtain

$$\text{resistance} = \int \frac{dx}{\pi y^2}\left\{1 + \frac{1}{2}y'^2 - \frac{(3y'^2 - yy'')^2}{48}\right\} \quad \ldots\ldots (11).$$

where, as before, accents denote differentiation with respect to x.

In a memoir on the Theory of Resonance published in the *Phil. Trans.* for 1871,* I had occasion to consider this problem, and I then proved that the first term of (11) represents an *inferior limit* to the resistance, and that the first and second terms together give a *superior limit*, these limits being applicable without any restriction on the form of the symmetrical boundary. It now appears that the superior limit coincides with the accurate value as far as a *second approximation*, and that the correction is approximately given by the third term in (11), which is essentially negative.

Other problems in Elecricity and Heat depending upon the potential may be treated by the above method. For example, the conductivity between the plane $z = 0$, and a neighbouring equipotential surface $z = z_1$, where z_1 is a function of x and y differing from a constant over a finite region only, is approximately

$$\iint \frac{dx\,dy}{z_1}\left\{1 + \frac{1}{3}\left(\frac{dz_1}{dx}\right)^2 + \frac{1}{3}\left(\frac{dz_1}{dy}\right)^2\right\} \quad \ldots\ldots\ldots\ldots(12),$$

* See also Maxwell's Electricity and Magnetism, Vol. I., Art. 307. In connection with this subject I may be allowed here to record that I have proved that the "correction for the end" of a cylindrical conductor is less than ·8242 of the radius, by a calculation based on an assumed motion containing two arbitrary constants; this result is probably very near the truth. The corresponding superior limit obtained from one arbitrary constant is ·8281 (Phil. Mag., Nov. 1872). If the motion at the end be assumed to be uniform, we get ·849 (Maxwell, Art. 309). An inferior limit is ·785.

where the area of integration includes the whole of the region through which z_1 varies.*

Finally it may be noticed that the preceding methods are applicable in two dimensions, when we replace x and y by any conjugate functions α, β of xy. By this transformation the scope of the analysis may be considerably increased, but to enlarge upon this would take us too far from the principal subject of the paper.

The Solution of Partial Differential Equations of the Second Order, with any number of variables, when there is a general first integral. *By* H. W. Lloyd Tanner, M.A.

[*Read 13th January*, 1876.]

In the first part of this paper it is proved that any equation of the second order which has a first integral

$$F\{u_1, \ldots\ldots u_n\} = 0,$$

consists of $\frac{1}{2} \cdot \frac{\lfloor 2n}{\{\lfloor n\}^2} + 2^{n-1}$ terms. One factor of each of these is the determinant

$$\left(s_{ij} \equiv \frac{d^2 z}{dx_i dx_j}\right)$$

$$\begin{vmatrix} s_{11}, s_{12} \ldots\ldots s_{1n} \\ s_{12}, s_{22} \ldots\ldots \\ \ldots\ldots\ldots\ldots \\ s_{1n} \ldots\ldots\ldots\ldots s_{nn} \end{vmatrix}$$

or a minor of this determinant. The other factor is a function of the derivatives of $u_1 \ldots u_n$, whose form is specified for each term.

In the second part we form $\frac{\lfloor 2n}{\lfloor n+1 \, \lfloor n-1}$ linear equations of the first order which serve to determine $u_1 \ldots u_n$. Of these equations n, and only n, are independent, and their coefficients are expressed directly or indirectly in terms of the coefficients of the given equation of the second order. There is always a second set of equations corresponding to another first integral; but, except possibly in one case, there are not more than two first integrals. The general theory is then applied to the case of two variables, and the results agree with those given by Boole; but with the advantage that here there are no irrelevant equations to be removed, and there is no need to consider special cases of some of the coefficients vanishing.

In the third part we consider the theory of the second integration.

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* Math. Tripos, 1876, Jan. 21, 1½ to 4, Question x.

If there be only one first integral (as distinguished from two identical first integrals) we can only obtain as many particular solutions as we please.

If two first integrals occur, it is shown that the arguments of the one furnish the equations required for the integration of the other. But in this case it is suggested that the equation should be solved by the method employed by Imschenetsky in the case of two variables.

If the two first integrals be identical, the complete primitive is obtained by equating to constants the different arguments of F; hence finding the values of $p_1 \dots p_n$, and integrating the expression

$$dz - p_1 dx_1 \dots - p_n dx_n,$$

which is then an exact differential.

The dependent variable being z, the n independent variables $x_1 \dots x_n$, we shall write p_i for $\frac{dz}{dx_i}$, and s_{ij} for $\frac{d^2 z}{dx_i dx_j}$.

By $\frac{du}{dx_i}$ we shall indicate the result of differentiating u with respect to x_i explicitly involved in u; by $\left(\frac{du}{dx_i}\right)$ we indicate that the differentiation is with respect to x_i as it occurs explicitly, and implicitly in z; in $\frac{d \cdot u}{dx_i}$ (a notation suggested by Imschenetsky *) the differentiation is with respect to x_i, however it occurs. Thus, if u be a function of $\dots p_j \dots x_i \dots z$, we should write

$$\frac{d \cdot u}{dx_i} \equiv \dots + \frac{du}{dp_j} s_{ij} + \dots + \left(\frac{du}{dx_i}\right)$$
$$\equiv \dots\dots\dots\dots + p_i \frac{du}{dz} + \frac{du}{dx_i}.$$

I. *On the Genesis of the Equation.*

(1.) Let $u_1 \dots u_n$ represent n mutually independent functions of $x_1 \dots x_n, z, p_1 \dots p_n$. Then, representing by F an arbitrary function, the general form of a first integral may be written

$$\mathrm{F}(u_1, \dots u_n) = 0 \dots\dots\dots\dots\dots\dots\dots\dots (1).$$

Differentiate this with respect to $x_1 \dots x_n$, and from the n equations thus formed eliminate the arbitrary functions $\frac{d\mathrm{F}}{du_1} \dots \frac{d\mathrm{F}}{du_n}$ ..., which are equivalent to $n-1$ only since the equations are homogeneous. The resulting equation will be the general form of the equation of the second order which admits of a first integral (1). This form is

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* And De Morgan, *Diff. and Int. Calc.*, p. 90.

$$\begin{vmatrix} \frac{d\,.\,u_1}{dx_1}, & \frac{d\,.\,u_1}{dx_2}, \ldots & \frac{d\,.\,u_1}{dx_n} \\ \frac{d\,.\,u_2}{dx_1}, & \ldots\ldots\ldots\ldots & \\ \ldots\ldots\ldots\ldots\ldots & & \\ \frac{d\,.\,u_n}{dx_1} \ldots\ldots & & \frac{d\,.\,u_n}{dx_n} \end{vmatrix} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

where $\frac{d\,.\,u_k}{dx_l} = \frac{du_k}{dp_1}\,.\,s_{1l} + \frac{du_k}{dp_2}\,.\,s_{2l} + \ldots\ldots + \frac{du_k}{dp_n}\,.\,s_{nl} + \left(\frac{du_k}{dx_l}\right)$.

(2.) In order to reduce this determinant to a more convenient form, we make use of the following generalization of the theorem implied in the rule for the multiplication of determinants. If

$$c_{kl} = \sum_{i=1}^{i=m} a_{ki}\,.\,b_{il},$$

the determinant $(c_{11} \ldots c_{nn})$ vanishes when m is less than n; but if m be not less than n its value is

$$\Sigma\,(a_{1i} \ldots a_{nj})\,(b_{i1} \ldots b_{jn}) \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

where the Σ extends to all the different products that can be formed by giving the n quantities $i \ldots j$, n different values between 1 and m inclusive. Hence the number of terms is $\frac{\lfloor m}{\lfloor n \,\lfloor m-n}$, which, when $m = n$ reduces to 1.

The determinant on the left of (2) is a particular case of this. For, i, j being any numbers not greater than n, it is only necessary to put

$$a_{ij} = \frac{du_i}{dp_j}, \quad b_{ij} = s_{ij},$$

and $a_{i,n+1}\,b_{n+1,j} + \ldots + a_{i,n+j}\,b_{n+j,j} + \ldots + a_{i,m}\,b_{m,j} = \left(\frac{du_i}{dx_j}\right)$,

for all values of i, j to make the two determinants identical. Now the last condition is satisfied identically if we assume

$$a_{i,n+j} = \left(\frac{du_i}{dx_j}\right),$$

$$b_{n+i,j} = 0, \text{ unless } j = i \text{ when } b_{n+j,j} = 1.$$

The value of m must be $2n$. It cannot be less, for we require the equation $a_{i,2n} = \left(\frac{du_i}{dx_n}\right)$. It cannot be more, for all the b's of the form $b_{2n+i,j}$ vanish identically in virtue of the last condition, and the fact that j is less than n. Hence in the expansion of (2) there will be $\frac{\lfloor 2n}{\{\lfloor n\}^2}$ terms.

(3.) Let us now seek to determine the constitution of these terms. From the theorem quoted it is clear that each term will be the product

of two determinants. The first or a-determinant will involve only the derivatives of $u_1 \dots u_n$. We shall write these determinants in the form $\frac{d \cdot (u_1 \dots \underline{u_j} \dots u_n)}{d \cdot (p_1 \dots (x_i) \dots p_n)}$, where the x is put in brackets to indicate that the symbols $\left(\frac{du}{dx_i}\right)$ are used, and it is placed under the u_j to show that the differentials $\frac{du}{dp_j}$ do not occur, $\left(\frac{du}{dx_i}\right)$ taking their place. We shall occasionally find it convenient to write the determinants in the form $\left|\begin{matrix} j \dots \\ \iota \dots \end{matrix}\right|$ the upper line showing the p's which do not occur, the lower line the x's by which they are respectively replaced, the original determinant being $\frac{d\,(u_1 \dots u_i \dots u_n)}{d\,(p_1 \dots p_i \dots p_n)}$.

(4.) The second factor of each term is a determinant involving only the quantities ..., ε_{ij} ..., unity and zero.

To determine the general form we may write down such a term as the following:—

$$\frac{d\,(u_1 \dots u_j \dots u_n)}{d\,(p_1 \dots (x_i) \dots p_n)} \times \begin{vmatrix} \varepsilon_{11}, \dots \varepsilon_{1i} \dots \varepsilon_{1n} \\ \dots\dots\dots\dots\dots \\ \varepsilon_{j-1,1} \dots \varepsilon_{j-1}, \dots \\ 0 \dots 1 \dots 0 \\ \dots\dots\dots\dots\dots \\ \varepsilon_{n,1} \dots \varepsilon_{ni} \dots \varepsilon_{nn} \end{vmatrix}$$

the second factor being written down by a reference to (3) and the values subsequently assigned to the b's. In this second factor remove the ι^{th} column to the front and then the j^{th} row to the top. It is, therefore,

$$(-)^{\iota+j} \begin{vmatrix} 1, & 0, & 0, & \dots 0 \\ \varepsilon_{1\iota}, & \varepsilon_{11}, & \varepsilon_{12}, & \dots \varepsilon_{1n} \\ \varepsilon_{2\iota}, & \varepsilon_{21}, & \varepsilon_{22} & \\ \vdots & \vdots & \vdots & \ddots \\ \vdots & \vdots & \vdots & \qquad \varepsilon_{nn} \end{vmatrix}$$

which is the minor of $(\varepsilon_{11} \dots \varepsilon_{nn})$ formed by erasing the ι^{th} column and j^{th} row, and affected with the sign $(-)^{\iota+j}$.

Again, suppose the first factor is $\left|\begin{matrix} j\ k \\ \iota\ \kappa \end{matrix}\right|$ or

$$\frac{d\,(u_1 \dots \underline{u_j} \dots \underline{u_k} \dots u_n)}{d\,(p_1 \dots (x_\iota) \dots (x_\kappa) \dots p_n)},$$

where we assume $k > j$, $\kappa > \iota$. In the second the factor the $\left.\begin{matrix} j \\ k \end{matrix}\right\}^{\text{th}}$ row vanishes, with the exception of the quantity common to it and the

$\left.\begin{matrix}\iota\\ \kappa\end{matrix}\right\}^{\text{th}}$ column, which is unity. By bringing these rows and columns to the edges, we find that the second factor is the minor of $(s_{11} \dots s_{nn})$ formed by erasing the rows j, k, and the columns ι, κ, and affected with the sign $(-)^{\iota+\kappa+j+k}$. The assumption as to the order of j, k, ι, κ does not affect this result; for interchanging $\iota\ \kappa$ (say) is simply equivalent to interchanging two columns in each of the determinants; viz., it changes the sign only of both factors, so that the product is unaltered.

This process is clearly general. Hence the s-factors in the expansion of (2) are the minors that can be obtained from $(s_{11} \dots s_{nn})$ including this determinant itself and unity as extremes. The coefficient of the minor formed by erasing the rows j, k, l ..., and the columns ι, κ, λ ... is the Jacobian

$$(-)^{\iota+\kappa+\lambda\dots+i+j+k-\dots} \left| \begin{matrix} j, & k, & l, & \dots \\ \iota, & \kappa, & \lambda, & \dots \end{matrix} \right|,$$

or

$$(-)^{\dots\dots} \frac{d\,(\dots u_j \dots u_k \dots u_l \dots)}{d\,(\dots(x_\iota)\dots(x_\kappa)\dots(x_\lambda)\dots)},$$

where j, k, l ... and ι, κ, λ ... are in ascending order of magnitude.

It is easy to verify that this gives the same number of terms as we obtained before. For there are $\left\{\frac{\lfloor n}{\lfloor n-r \lfloor r}\right\}^2$ minors of the r^{th} order, since these may be formed by retaining any r rows and any r columns of the n; and

$$1^2+n^2+\left\{\frac{n(n-1)}{1\,.\,2}\right\}^2+\dots\dots+n^2+1^2=\frac{\lfloor 2n}{\{\lfloor n\}^2},$$

which agrees with the result given at the conclusion of Art. 2.

(5.) Let us now take account of the fact that $s_{ij}=s_{ji}$, or that the determinant $(s_{11} \dots s_{nn})$ is symmetrical. The symmetrical minors of the r^{th} order are of the form $(s_{ii}, \dots s_{kk} \dots s_{ll})$, and there are $\frac{\lfloor n}{\lfloor r \lfloor n-r}$ of them, since any combination r together of the n suffixes may be taken. Altogether, then, there are $1+n+\frac{n\,(n-1)}{1\,.\,2}+\dots+n+1$, or 2^n symmetrical minors. The coefficients of these are of the form

$$\frac{d\,(\dots u_j \dots u_k \dots)}{d\,(\dots(x_j)\dots(x_k)\dots)}.$$

The sign is $(-)^{2(j+k+\dots)}$ or positive.

All the other terms in the expansion of (2) involve unsymmetrical minors. These occur in pairs: for the minor formed by erasing the rows j, k, ... and the columns ι, κ, ... is identical with that formed by erasing the rows ι, κ, ... and the columns j, k, ...

Taking each pair together, we have $\frac{1}{2}\cdot\left\{\frac{\lfloor 2n}{\{\lfloor n\}^2}-2^n\right\}$ terms involving unsymmetrical minors of $(s_{11}\ldots s_{nn})$, and the coefficient of each will be of the form

$$(-)^{j+k+\ldots+s+\kappa}\left\{\frac{d\,(\ldots\ u_j\ \ldots\ u_k\ \ldots)}{d\,(\ldots\ (x_\iota)\ \ldots\ (x_\kappa)\ \ldots)}+\frac{d\,(\ldots\ u_\iota\ \ldots\ u_\kappa\ \ldots)}{d\,(\ldots\ (x_j)\ \ldots\ (x_k)\ \ldots)}\right\}.$$

(6.) These results enable us to write down the form of the equation of the second order which includes all that admit of a first integral of the form (1). For instance, in the case of three independent variables, this is

$$\mathrm{U}\begin{vmatrix} s_{11}, s_{12}, s_{13} \\ s_{12}, s_{22}, s_{23} \\ s_{13}, s_{23}, s_{33}\end{vmatrix}+\mathrm{U}_1\begin{vmatrix} s_{22}, s_{23} \\ s_{23}, s_{33}\end{vmatrix}+\mathrm{U}_2\begin{vmatrix} s_{11}, s_{13} \\ s_{13}, s_{33}\end{vmatrix}+\mathrm{U}_3\begin{vmatrix} s_{11}, s_{12} \\ s_{12}, s_{22}\end{vmatrix}$$

$$-2\mathrm{U}_{\substack{2\\3}}\begin{vmatrix} s_{11}, s_{12} \\ s_{13}, s_{23}\end{vmatrix}+2\mathrm{U}_{\substack{1\\3}}\begin{vmatrix} s_{12}, s_{22} \\ s_{13}, s_{23}\end{vmatrix}-2\mathrm{U}_{\substack{1\\2}}\begin{vmatrix} s_{12}, s_{23} \\ s_{13}, s_{33}\end{vmatrix}$$

$$+\mathrm{U}_{23}.s_{11}+\mathrm{U}_{13}.s_{22}+\mathrm{U}_{12}.s_{33}-2\mathrm{U}_{\substack{12\\13}}.s_{23}+2\mathrm{U}_{\substack{12\\23}}.s_{31}-2\mathrm{U}_{\substack{13\\23}}.s_{12}+\mathrm{U}_{123}=0\ldots(4),$$

where, if μ be any indeterminate multiplier,

$$\mathrm{U}=\mu\cdot\frac{d\,(u_1\,.\,u_2\,.\,u_3)}{d\,(p_1\,.\,p_2\,.\,p_3)},\quad \mathrm{U}_{123}=\mu\cdot\frac{d\,(u_1\,.\,u_2\,.\,u_3)}{d\,((x_1)\,.\,(x_2)\,.\,(x_3))},$$

$$\mathrm{U}_1=\mu\cdot\frac{d\,(u_1\,.\,u_2\,.\,u_3)}{d\,((x_1)\,.\,p_2\,.\,p_3)},\quad 2\mathrm{U}_{\substack{2\\3}}=\mu\left\{\frac{d\,(u_1\,.\,u_2\,.\,u_3)}{d\,(p_1\,.\,(x_3)\,.\,p_3)}+\frac{d\,(u_1\,.\,u_2\,.\,u_3)}{d\,(p_1\,.\,p_2\,.\,(x_2))}\right\},$$

$$\mathrm{U}_{23}=\mu\cdot\frac{d\,(u_1\,.\,u_2\,.\,u_3)}{d\,(p_1\,.\,(x_2)\,.\,(x_3))},$$

$$\mathrm{U}_{\substack{12\\13}}=\mu\left\{\frac{d\,(u_1\,.\,u_2\,.\,u_3)}{d\,((x_1)\,.\,(x_3)\,.\,p_3)}+\frac{d\,(u_1\,.\,u_2\,.\,u_3)}{d\,((x_1)\,.\,p_2\,.\,(x_2))}\right\},$$

$$\&c.=\&c.$$

That the given equation of the second order should be of the general form indicated by the equation (4), regarding therein the coefficients U as arbitrary, is necessary, but not sufficient to ensure the existence of a general first integral. In the case of n independent variables we should have $\frac{1}{2}\cdot\frac{\lfloor 2n}{\{\lfloor n\}^2}+2^{n-1}$ coefficients; but these are expressible in terms of $n+1$ independent quantities $u_1\ldots u_n$, μ. Thus the coefficients must satisfy

$$\frac{1}{2}\cdot\frac{\lfloor 2n}{\{\lfloor n\}^2}+2^{n-1}-n-1\ \text{conditions.}$$

II. *Determination of the First Integral.*

(7.) We now attack the problem converse to that discussed in the first part of this paper; viz., starting with a known equation of the second

order, we endeavour to find a first integral

$$F\{u_1 \ldots\ldots u_n\} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

when one exists. This will be accomplished by discovering the n quantities $u_1 \ldots u_n$.

Now the equation

$$\begin{vmatrix} \left(\frac{du}{dx_j}\right), & \frac{du}{dp_1} & \ldots\ldots & \frac{du}{dp_n} \\ \left(\frac{du_1}{dx_j}\right), & \frac{du_1}{dp_1} & \ldots\ldots & \frac{du_1}{dp_n} \\ \ldots\ldots & \ldots & \ldots\ldots & \ldots \\ \ldots\ldots & \ldots & \ldots\ldots & \ldots \\ \left(\frac{du_n}{dx_j}\right), & \frac{du_n}{dp_1} & \ldots\ldots & \frac{du_n}{dp_n} \end{vmatrix} = 0 \ldots\ldots\ldots\ldots\ldots (5)$$

is evidently satisfied when $u = u_1, \ldots$ or u_n; for then two rows of the determinant become identical.

In this equation the differentiations are performed with respect to $n+1$ only of the $2n$ quantities $x_1 \ldots x_n, p_1 \ldots p_n$; and since we may form a similar equation, using *any* $n+1$ of these quantities, there are evidently $\frac{\lfloor 2n}{\lfloor n+1 \, \lfloor n-1}$ equations of the same type as (5).

(8.) The equation (5) may be written in a more convenient form. It is evidently homogeneous of the first degree in the derivatives of u. The coefficient of $\left(\frac{du}{dx_j}\right)$ is the Jacobian $\frac{d\,(u_1 \ldots u_n)}{d\,(p_1 \ldots p_n)}$. The coefficient of $\frac{du}{dp_r}$, as may be seen by interchanging the columns containing $\left(\frac{du}{dx_j}\right)$, $\frac{du}{dp_r}$, is the same Jacobian with $\left(\frac{du}{dx_j}\right)$, $\left(\frac{du_2}{dx_j}\right)\ldots$ in the place of $\frac{du_1}{dp_r}$, $\frac{du_2}{dp_r}$, ..., and affected with a negative sign. We shall represent, as before, the result of this substitution by the symbol $\left| \begin{matrix} r \\ j \end{matrix} \right|$, the Jacobian $\frac{d\,(u_1 \ldots u_n)}{d\,(p_1 \ldots p_n)}$ being accordingly represented by the symbol $\left| \quad \right|$. The equation (5) may then be written in the form

$$\left| \quad \right| + \left(\frac{du}{dx_j}\right) - \Sigma_r \left| \begin{matrix} r \\ j \end{matrix} \right| \frac{du}{dp_r} = 0 \ldots\ldots\ldots\ldots (6).$$

Concerning the $\frac{\lfloor 2n}{\lfloor n+1 \, \lfloor n-1}$ equations of the form (5) or (6), we shall now prove:—that n of them are independent; that not more than n are independent; and that the coefficients can be expressed in terms of the coefficients of the given equation of the second order.

(9.) *There are n independent equations.* From (3) we can obtain n equations by putting $j = 1, \ldots n$, in succession. Now, if $\frac{d\,(u_1 \ldots u_n)}{d\,(p_1 \ldots p_n)}$ does not vanish, these equations will be algebraically independent, for each will contain a quantity $\left(\frac{du}{dx_1}\right)$, $\left(\frac{du}{dx_2}\right)$ ..., which occurs in none of other equations of the system. But the marked minor in (5) may be replaced by any of the Jacobians of $u_1 \ldots u_n$, and if the one selected does not vanish, we get n independent equations by using for the first column differentials with respect to the n quantities that do not occur in the Jacobian. Such a system is always obtainable, since, if all the Jacobians of $u_1 \ldots u_n$ vanish, the same is true of the coefficients of the equation of the second order by the first part of this paper.

(10.) *Only n of the equations are independent.* For we have a matrix, $|\,a_{i,j}\,|$, $n+1$ deep by $m+n$ broad (here $m=n$). The m determinants obtained by taking in succession each of the first m columns with the last n columns all vanish: *i. e.*, A_1, A_2 ... A_{n+1} being the $n+1$ determinants (of order n) of the last n columns, we have for all values of j,

$$A_1 a_{1,j} + A_2 a_{2,j} + \ldots + A_{n+1} a_{n+1,j} = 0.$$

But A_1, A_2 ... A_{n+1} do not all vanish (for the marked minor is to be a Jacobian which does not vanish): therefore all the determinants of the matrix vanish. Hence, if the n equations of Art. 9 be satisfied, the other equations of the system become identities.

(11.) If we express that the $\frac{\lfloor 2n}{\lfloor n+1 \,\lfloor n-1}$ equations for u are algebraically equivalent to n only, we obtain a remarkable series of theorems which enable us to express the coefficients of these equations in terms of the coefficients of the given equation of the second order, and which give implicitly some of the necessary relations between the coefficients of the given equation.

For instance, supposing that $\frac{d\,(u_1 \ldots u_n)}{d\,(p_1 \ldots p_n)}$ or $\left|\ \right|$ does not vanish, let us eliminate $\frac{du}{dp_\kappa}$ from the two equations

$$\left|\ \right| \cdot \left(\frac{du}{dx_j}\right) - \Sigma_r \left|\begin{matrix} r \\ j \end{matrix}\right| \frac{du}{dp_r} = 0,$$

$$\left|\ \right| \cdot \left(\frac{du}{dx_k}\right) - \Sigma_r \left|\begin{matrix} r \\ k \end{matrix}\right| \frac{du}{dp_r} = 0.$$

The result is

$$\left|\ \right| \times \left|\begin{matrix} \kappa \\ k \end{matrix}\right| \times \left(\frac{du}{dx_j}\right) - \left|\ \right| \times \left|\begin{matrix} \kappa \\ j \end{matrix}\right| \times \left(\frac{du}{dx_k}\right) - \Sigma_r \begin{vmatrix} \left|\begin{matrix} r \\ j \end{matrix}\right|, & \left|\begin{matrix} r \\ k \end{matrix}\right| \\ \left|\begin{matrix} \kappa \\ j \end{matrix}\right|, & \left|\begin{matrix} \kappa \\ k \end{matrix}\right| \end{vmatrix} \frac{du}{dp_r} = 0.$$

Now this must be identical with the equation formed from the determinant whose first row is

$$\left(\frac{du}{dx_j}\right),\ \frac{du}{dp_1} \cdots\cdots \frac{du}{dp_{\kappa-1}},\ \left(\frac{du}{dx_k}\right), \cdots\cdots,$$

or
$$\left|\begin{matrix}\kappa\\ k\end{matrix}\right| \times \left(\frac{du}{dx_j}\right) - \left|\begin{matrix}\kappa\\ j\end{matrix}\right| \times \left(\frac{du}{dx_k}\right) - \Sigma_r \left|\begin{matrix}r\ \kappa\\ j\ k\end{matrix}\right| \cdot \frac{du}{dp_r} = 0;$$

for otherwise more than n equations of the system would be independent.

Comparing coefficients, we get, taking $r = \iota$,

$$\left|\begin{matrix}\left|\begin{matrix}\iota\\ j\end{matrix}\right|, & \left|\begin{matrix}\iota\\ k\end{matrix}\right| \\ \left|\begin{matrix}\kappa\\ j\end{matrix}\right|, & \left|\begin{matrix}\kappa\\ k\end{matrix}\right|\end{matrix}\right| = \left|\ \right| \times \left|\begin{matrix}\iota\ \kappa\\ j\ k\end{matrix}\right| \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (7),$$

where in analogy with the previous notation we write $\left|\begin{matrix}\iota\ \kappa\\ j\ k\end{matrix}\right|$ for the result of substituting $\left(\frac{du_1}{dx_j}\right)$ ... and $\left(\frac{du_1}{dx_k}\right)$ ... for $\frac{du_1}{dp_\iota}$... and $\frac{du_1}{dp_\kappa}$ ... respectively in $|\ |$.

Again, from the three equations

$$\left|\ \right| \left(\frac{du}{dx_j}\right) - \Sigma_r \left|\begin{matrix}r\\ j\end{matrix}\right| \frac{du}{dp_r} = 0,$$

$$\left|\ \right| \left(\frac{du}{dx_k}\right) - \Sigma_r \left|\begin{matrix}r\\ k\end{matrix}\right| \frac{du}{dp_r} = 0,$$

$$\left|\ \right| \left(\frac{du}{dx_l}\right) - \Sigma_r \left|\begin{matrix}r\\ l\end{matrix}\right| \frac{du}{dp_r} = 0,$$

eliminating $\frac{du}{dp_\kappa}$, $\frac{du}{dp_\lambda}$, we obtain the equation

$$\left|\ \right| \times \left\{ \left(\frac{du}{dx_j}\right) \left|\begin{matrix}\left|\begin{matrix}\kappa\\ k\end{matrix}\right|, & \left|\begin{matrix}\kappa\\ l\end{matrix}\right| \\ \left|\begin{matrix}\lambda\\ k\end{matrix}\right|, & \left|\begin{matrix}\lambda\\ l\end{matrix}\right|\end{matrix}\right| - \left(\frac{du}{dx_k}\right) \left|\begin{matrix}\left|\begin{matrix}\kappa\\ j\end{matrix}\right|, & \left|\begin{matrix}\kappa\\ l\end{matrix}\right| \\ \left|\begin{matrix}\lambda\\ j\end{matrix}\right|, & \left|\begin{matrix}\lambda\\ l\end{matrix}\right|\end{matrix}\right| + \left(\frac{du}{dx_l}\right) \left|\begin{matrix}\left|\begin{matrix}\kappa\\ j\end{matrix}\right|, & \left|\begin{matrix}\kappa\\ k\end{matrix}\right| \\ \left|\begin{matrix}\lambda\\ j\end{matrix}\right|, & \left|\begin{matrix}\lambda\\ k\end{matrix}\right|\end{matrix}\right| \right\}$$

$$- \Sigma_r \left|\begin{matrix}\left|\begin{matrix}r\\ j\end{matrix}\right|, & \left|\begin{matrix}r\\ k\end{matrix}\right|, & \left|\begin{matrix}r\\ l\end{matrix}\right| \\ \left|\begin{matrix}\kappa\\ j\end{matrix}\right|, & \left|\begin{matrix}\kappa\\ k\end{matrix}\right|, & \left|\begin{matrix}\kappa\\ l\end{matrix}\right| \\ \left|\begin{matrix}\lambda\\ j\end{matrix}\right|, & \left|\begin{matrix}\lambda\\ k\end{matrix}\right|, & \left|\begin{matrix}\lambda\\ l\end{matrix}\right|\end{matrix}\right| \frac{du}{dp_r} = 0.$$

The first three terms of this equation reduce by (7) to

$$\left\{ \left|\ \right| \right\}^2 \times \left\{ \left(\frac{du}{dx_j}\right) \left|\begin{matrix}\kappa\ \lambda\\ k\ l\end{matrix}\right| - \left(\frac{du}{dx_k}\right) \left|\begin{matrix}\kappa\ \lambda\\ j\ l\end{matrix}\right| - \left(\frac{du}{dx_l}\right) \left|\begin{matrix}\kappa\ \lambda\\ k\ j\end{matrix}\right| \right\}.$$

Compare this with the equation derived from the determinant whose top row is

$$\left(\frac{du}{dx_j}\right),\ \frac{du}{dp_1},\ \dots\ \frac{du}{dp_{\kappa-1}},\ \left(\frac{du}{dx_k}\right),\ \dots\ \frac{du}{dp_{\lambda-1}},\ \left(\frac{du}{dx_l}\right),\ \dots\ \frac{du}{dp_n};$$

viz., with the equation

$$\left|\begin{matrix}\kappa\,\lambda\\ k\,l\end{matrix}\right|\left(\frac{du}{dx_j}\right)-\left|\begin{matrix}\kappa\,\lambda\\ j\,l\end{matrix}\right|\left(\frac{du}{dx_k}\right)-\left|\begin{matrix}\kappa\,\lambda\\ k\,j\end{matrix}\right|\left(\frac{du}{dx_l}\right)-\Sigma_r\left|\begin{matrix}r\,\kappa\,\lambda\\ j\,k\,l\end{matrix}\right|\frac{du}{dp_r}=0;$$

and we obtain the relation $(r=\iota)$

$$\left|\begin{matrix}\left|\begin{matrix}\iota\\ j\end{matrix}\right|, & \left|\begin{matrix}\iota\\ k\end{matrix}\right|, & \left|\begin{matrix}\iota\\ l\end{matrix}\right|\\ \left|\begin{matrix}\kappa\\ j\end{matrix}\right|, & \left|\begin{matrix}\kappa\\ k\end{matrix}\right|, & \left|\begin{matrix}\kappa\\ l\end{matrix}\right|\\ \left|\begin{matrix}\lambda\\ j\end{matrix}\right|, & \left|\begin{matrix}\lambda\\ k\end{matrix}\right|, & \left|\begin{matrix}\lambda\\ l\end{matrix}\right|\end{matrix}\right| = \left\{\left|\begin{matrix}\ \\ \ \end{matrix}\right|\right\}^2\times\left|\begin{matrix}\iota\,\kappa\,\lambda\\ j\,k\,l\end{matrix}\right| \quad \dots\dots (8).$$

By an inductive proof precisely similar to the above, we infer generally that

$$\left|\begin{matrix}\left|\begin{matrix}\iota\\ j\end{matrix}\right|, & \left|\begin{matrix}\iota\\ k\end{matrix}\right|, & \dots\\ \left|\begin{matrix}\kappa\\ j\end{matrix}\right|, & \left|\begin{matrix}\kappa\\ k\end{matrix}\right|, & \dots\\ \vdots & \vdots & \end{matrix}\right| = \left\{\left|\begin{matrix}\ \\ \ \end{matrix}\right|\right\}^{t-1}\times\left|\begin{matrix}\iota\,\kappa\,\dots\\ j\,k\,\dots\end{matrix}\right| \quad \dots\dots (9),$$

the determinant on the left being of the t^{th} order.

More generally, starting with the determinant $\left|\begin{matrix}a\ \dots\ \beta\\ a\ \dots\ b\end{matrix}\right|$ in the place of the marked minor in (5), we get a series of theorems of the form

$$\left|\begin{matrix}\left|\begin{matrix}a\ \dots\ \beta\iota\\ a\ \dots\ bj\end{matrix}\right|, & \left|\begin{matrix}a\ \dots\ \beta\iota\\ a\ \dots\ bk\end{matrix}\right|, & \dots\\ \left|\begin{matrix}a\ \dots\ \beta\kappa\\ a\ \dots\ bj\end{matrix}\right|, & \left|\begin{matrix}a\ \dots\ \beta_\kappa\\ a\ \dots\ bj\end{matrix}\right|, & \dots\\ \vdots & \vdots & \end{matrix}\right| = \left\{\left|\begin{matrix}a\ \dots\ \beta\\ a\ \dots\ b\end{matrix}\right|\right\}^{t-1}\times\left|\begin{matrix}a\ \dots\ \beta\iota\kappa\ \dots\\ a\ \dots\ bjk\ \dots\end{matrix}\right| \quad \dots\dots (10),$$

t being the number of quantities $\iota, \kappa, \dots$.

The proof we have given of these theorems ceases to be rigorous when $\left|\ \right|$ or $\left|\begin{matrix}a\ \dots\ \beta\\ a\ \dots\ b\end{matrix}\right|$ respectively vanish. But the truth of these theorems evidently depends upon the manner in which the derivatives of $u_1 \dots u_n$ are involved, and not upon their magnitudes, and it is therefore probable that they remain true even in this case.

It is obvious that we can obtain fresh series of theorems either by expanding (9) or (10) in products of conjugate minors and replacing these by their values found from the same equations, or we may start

from a fresh set of equations and combine them in various ways. Thus, for example, multiplying $n+1$ equations by $n+1$ indeterminate multipliers and eliminating these multipliers between the $2n$ equations formed by equating to zero the coefficient of each term in the sum, we shall obtain n identical relations.

(12.) Supposing, as before, that $||$ does not vanish, let us, in (7), put $\iota = j$, $\kappa = k$, and write the determinant at length. Thus we get

$$\left|\begin{matrix} j \\ k \end{matrix}\right| \cdot \left|\begin{matrix} k \\ j \end{matrix}\right| = \left|\begin{matrix} j \\ j \end{matrix}\right| \cdot \left|\begin{matrix} k \\ k \end{matrix}\right| - \left|\right| \cdot \left|\begin{matrix} jk \\ jk \end{matrix}\right| \quad \text{.....(11).}$$

Now the determinants on the right are known, for they are the coefficients (divided by μ) of symmetrical minors of $(s_{11} \dots s_{nn})$ in the given equation of the second order. Also the sum of $\left|\begin{matrix} j \\ k \end{matrix}\right|, \left|\begin{matrix} k \\ j \end{matrix}\right|$ is known, since it is the coefficient of a certain unsymmetrical minor of $(s_{11} \dots s_{nn})$ divided by μ. Hence $\mu \left|\begin{matrix} j \\ k \end{matrix}\right|, \mu \left|\begin{matrix} k \\ j \end{matrix}\right|$ are two roots of a certain quadratic equation whose coefficients are given in terms of the coefficients of the equation of the second order to be solved.

For example, in the equation with three independent variables given in (4) we find that $\mu \left|\begin{matrix} 2 \\ 3 \end{matrix}\right|, \mu \left|\begin{matrix} 3 \\ 2 \end{matrix}\right|$ are the two roots of the quadratic

$$\xi^2 - 2U_{\substack{2 \\ 3}} \cdot \xi + U_2 U_3 - U U_{23} = 0.$$

Again, supposing that $\left|\begin{matrix} a \dots \beta \\ a \dots b \end{matrix}\right|$ does not vanish, we obtain from (10), on putting $t = 2$, $\iota = j$, $\kappa = k$, the equation

$$\left|\begin{matrix} a \dots \beta j \\ a \dots b k \end{matrix}\right| \left|\begin{matrix} a \dots \beta k \\ a \dots b j \end{matrix}\right| = \left|\begin{matrix} a \dots \beta j \\ a \dots b j \end{matrix}\right| \times \left|\begin{matrix} a \dots \beta k \\ a \dots b k \end{matrix}\right| - \left|\begin{matrix} a \dots \beta \\ a \dots b \end{matrix}\right| \times \left|\begin{matrix} a \dots \beta j k \\ a \dots b j k \end{matrix}\right|;$$

whence, if $a = a$, $\dots$ $b = \beta$, we can find the two factors on the left.

Thus, in the equation (4), the quantities $\mu \left|\begin{matrix} 1\,2 \\ 1\,3 \end{matrix}\right|, \mu \left|\begin{matrix} 1\,3 \\ 1\,2 \end{matrix}\right|$ are the roots of the quadratic equation

$$\xi^2 - 2U_{\substack{12 \\ 13}} \cdot \xi + U_{12} \cdot U_{13} - U_1 \cdot U_{123} = 0.$$

The occurrence of an indeterminate factor μ in these determinations is immaterial; for the equations for u are homogeneous in μ after substitution, so that this factor divides out.

(13.) The determination of the values of, say, $\left|\begin{matrix} j \\ k \end{matrix}\right|, \left|\begin{matrix} k \\ j \end{matrix}\right|$, as far as it is indicated above, leaves us free to assign to each of these functions one of a pair of values; and it is now necessary to examine if this ambiguity is limited in any way. In the first place, it is evident that when we have assigned values to all the determinants $\left|\begin{matrix} j \\ k \end{matrix}\right|$, &c., or $\left|\begin{matrix} a \dots bj \\ a \dots bk \end{matrix}\right|$,

&c., that the others are determinate, supposing that $\left|\ \right|$ or $\left|\begin{matrix} a \dots b \\ a \dots b \end{matrix}\right|$ respectively do not vanish. In fact, a single value of each is given by (9) or (10). We shall now show that in general, when $\left|\begin{matrix} j \\ k \end{matrix}\right| \cdot \left|\begin{matrix} k \\ j \end{matrix}\right|$ have values assigned to them, all the Jacobians $\left|\begin{matrix} a \\ a \end{matrix}\right|$ will admit of only single values.

In (8) put $\iota=j$, $\kappa=k$, $\lambda=l$. Expand the determinant and transpose to the right side all the terms such as $\left|\begin{matrix} j \\ j \end{matrix}\right| \left|\begin{matrix} k \\ k \end{matrix}\right| \left|\begin{matrix} l \\ l \end{matrix}\right|$, $\left|\begin{matrix} j \\ j \end{matrix}\right| \left|\begin{matrix} k \\ l \end{matrix}\right| \left|\begin{matrix} l \\ k \end{matrix}\right|$, whose values are free from ambiguity. Thus we find that

$$\left|\begin{matrix} j \\ k \end{matrix}\right| \left|\begin{matrix} k \\ l \end{matrix}\right| \left|\begin{matrix} l \\ j \end{matrix}\right| + \left|\begin{matrix} j \\ l \end{matrix}\right| \left|\begin{matrix} k \\ j \end{matrix}\right| \left|\begin{matrix} l \\ k \end{matrix}\right|$$

has a determinate value for each value of l.

Now these equations, and others which we obtain by putting $t=4$, 5, &c. in (9), are not altered if we change all the quantities $\left|\begin{matrix} j \\ l \end{matrix}\right|$, &c. into the quantities $\left|\begin{matrix} l \\ j \end{matrix}\right|$; but they generally cease to be satisfied when some only of these quantities are thus interchanged. Thus from the equations (6) we can always form a second set of equations, the solutions of which will give a second first integral; and in general there are only two such sets of equations.

I have attempted to form equations such as to admit of more than one pair of first integrals, by making for all values of l

$$\left|\begin{matrix} k \\ l \end{matrix}\right| \left|\begin{matrix} l \\ j \end{matrix}\right| = \left|\begin{matrix} j \\ l \end{matrix}\right| \left|\begin{matrix} k \\ j \end{matrix}\right|.$$

In all cases I have found, however, that, though two pairs of integrals were obtained, yet in virtue of these very conditions they were identical; but I am not prepared to prove that such is always the case.

(14.) The theorems (9), (10), and others allied to them, serve also to indicate the necessary relations existing between the coefficients of the given equation of the second order. In fact, if we express these coefficients in terms of Jacobians of $u_1 \dots u_n$, as in the first part, we shall have a system of equations between which we can eliminate u_1, &c., leaving a series of necessary relations between the original coefficients. But besides these relations, which do not involve the derivatives of the coefficients, there are also others which are necessary to insure that the n equations (6) should have n common integrals. These conditions are very complex in form and do not invite a direct investigation.

It may, however, be noted that one of the conditions that the n

equations should have $n+1$ common integrals is that $\left|\begin{matrix}k\\j\end{matrix}\right| = \left|\begin{matrix}j\\k\end{matrix}\right|$ for all values of j, k. This implies that the quantity

$$dz - p_1\,dx_1 \ldots - p_n\,dx_n,$$

which is the differential of the $n+1^{\text{th}}$ integral, is in this case an exact differential; a result which will be verified hereafter.

But the expression of these conditions is fortunately not necessary for our purpose. To find a first integral we take any set of n equations (5) which are independent. If these have not n common integrals there is no first integral. If they have, substitute them for $u_1 \ldots u_n$ in (1), and find if the equation is satisfied by this integral. It is generally convenient to render our trial integral as free from ambiguity as possible by means of the theorems (9) or (10) as explained above; but if this be not done a comparison with the original equation at once serves to remove all inadmissible ambiguity.

(15.) It may be well to illustrate the results obtained by an application of them to the general equation with two variables,

$$\mathrm{U}\,(rt-s^2)+\mathrm{R}r-2\mathrm{S}s+\mathrm{T}t+\mathrm{V}=0.$$

The arguments of the first integral of this equation satisfy the four equations

$$\mathrm{U}\left(\frac{du}{dx}\right) - \mathrm{T}\,\frac{du}{dp} - m_1\frac{du}{dq} = 0 \quad \ldots\ldots\ldots\ldots\ldots (\alpha),$$

$$\mathrm{U}\left(\frac{du}{dy}\right) - m_2\frac{du}{dp} - \mathrm{R}\,\frac{du}{dq} = 0 \quad \ldots\ldots\ldots\ldots\ldots (\beta),$$

$$\mathrm{R}\left(\frac{du}{dx}\right) - m_1\left(\frac{du}{dy}\right) + \mathrm{V}\,\frac{du}{dp} = 0 \quad \ldots\ldots\ldots\ldots\ldots (\gamma),$$

$$m_2\left(\frac{du}{dx}\right) - \mathrm{T}\left(\frac{du}{dy}\right) - \mathrm{V}\,\frac{du}{dq} = 0 \quad \ldots\ldots\ldots\ldots\ldots (\delta),$$

where $m_1+m_2 = 2\mathrm{S}$.

These equations reduce to two, provided

$$m_1 m_2 = \mathrm{RT}-\mathrm{UV},$$

so that m_1, m_2 are the roots of

$$\xi^2-2\mathrm{S}\xi+\mathrm{RT}-\mathrm{UV}=0.$$

These results agree exactly with those given by Boole (Supp. Vol., chaps. xxviii., xxix.), making allowance for different notation. It may be interesting to notice that in chap. xxviii. Boole makes use of (α), (β) except when $\mathrm{U}=0$, when he employs (α) or (β) with (γ), and when $\mathrm{V}=0$, when either (γ) or (δ) is used with (β). In chap. xxix. equations (γ), (δ) are used; but the case of $\mathrm{V}=0$, when they become identical, is not specially mentioned.

III. *Determination of the Second Integral.*

(16.) In order to integrate the non-linear differential equation of the first order

$$F\{u_1 \dots u_n\} = 0 \quad \dots\dots\dots\dots\dots\dots (1),$$

we must find $n-1$ functions of p_1, &c. such that, v representing any one of these quantities,

$$\sum_{i=1}^{i=n} \left\{ \left(\frac{dF}{dx_i}\right) \frac{dv}{dp_i} - \frac{dF}{dp_i} \left(\frac{dv}{dx_i}\right) \right\} = 0 \quad \dots\dots\dots\dots (12).$$

Now, in general, the values of v will depend upon the particular form of F, and in this case the general method for obtaining a second integral is to assign to F a particular form and then integrate. We thus obtain as many particular solutions of the given equation of the second order as we please.

But it may happen that the equation (12) is satisfied independently of the form of F. In order that this may be so, we require that the coefficients of the arbitrary functions $\frac{dF}{du_1}$, $\frac{dF}{du_2}$, &c. should simultaneously vanish. This gives for v the n equations obtained by putting k successively equal to 1, 2 ... n in the equation

$$\sum_{i=1}^{i=n} \left\{ \left(\frac{du_k}{dx_i}\right) \frac{dv}{dp_i} - \left(\frac{dv}{dx_i}\right) \frac{du_k}{dp_i} \right\} = 0.$$

Now, if from these n equations we eliminate $n-1$ of the quantities $\frac{dv}{dp_i}$, $\left(\frac{dv}{dx_i}\right)$, say all the $\left(\frac{dv}{dx}\right)$'s except $\left(\frac{dv}{dx_j}\right)$, we get an equation such as

$$\left| \right| \times \left(\frac{dv}{dx_j}\right) - \Sigma \left| \begin{matrix} j \\ r \end{matrix} \right| \frac{dv}{dp_r} = 0.$$

But this equation is simply (6) with $\left| \begin{matrix} j \\ r \end{matrix} \right|$ substituted for $\left| \begin{matrix} r \\ j \end{matrix} \right|$, and we therefore conclude that, (1) being one of the first integrals of the given equation, the equations for integrating (1) are simply those which express that the arguments of the other first integral are constant; and in the particular case in which the two first integrals become identical, that the arguments $u_1 \dots u_n$ equated to constants will yield values of $p_1 \dots p_n$ which render

$$dz - p_1 dx_1 \dots - p_n dx_n$$

an exact differential, so that a complete primitive can be found.

(17.) As an example of the simpler case we take the equation

$$x_1 x_2 x_3 \begin{vmatrix} s_{11}, & s_{12}, & s_{13} \\ s_{12}, & s_{22}, & s_{23} \\ s_{13}, & s_{23}, & s_{33} \end{vmatrix} - p_1 x_2 x_3 \begin{vmatrix} s_{22}, & s_{23} \\ s_{23}, & s_{33} \end{vmatrix} - x_1 p_2 x_3 \begin{vmatrix} s_{11}, & s_{13} \\ s_{13}, & s_{33} \end{vmatrix}$$

$$- x_1 x_2 p_3 \begin{vmatrix} s_{11}, & s_{12} \\ s_{12}, & s_{22} \end{vmatrix}$$

$$+ x_1 p_2 p_3 s_{11} + p_1 x_2 p_3 s_{22} + p_1 p_2 x_3 s_{33} - p_1 p_2 p_3 = 0.$$

Since no unsymmetrical minors occur in this equation the sum of each pair of Jacobians such as $\left|\begin{matrix}1\\2\end{matrix}\right|$, $\left|\begin{matrix}2\\1\end{matrix}\right|$ vanishes. Also their product vanishes, that of $\left|\begin{matrix}1\\2\end{matrix}\right| \times \left|\begin{matrix}2\\1\end{matrix}\right|$, for example, being

$$p_1 x_2 x_3 \cdot x_1 p_2 x_3 - x_1 x_2 x_3 \cdot p_1 p_2 x_3.$$

Hence all such Jacobians vanish.

Now, forming the equations (6), we get

$$x_1 \left(\frac{du}{dx_1}\right) + p_1 \frac{du}{dp_1} = 0,$$

$$x_2 \left(\frac{du}{dx_2}\right) + p_2 \frac{du}{dp_2} = 0,$$

$$x_3 \left(\frac{du}{dx_3}\right) + p_3 \frac{du}{dp_3} = 0.$$

These give for a first integral

$$\mathrm{F}\left\{\frac{p_1}{x_1}, \frac{p_2}{x_2}, \frac{p_3}{x_3}\right\} = 0,$$

which satisfies the given equation. The other first integral is identical with this.

To find the complete primitive, put

$$\frac{p_1}{x_1} = 2a_1, \quad \frac{p_2}{x_2} = 2a_2, \quad \frac{p_3}{x_3} = 2a_3;$$

therefore $$dz = 2a_1 x_1 dx_1 + 2a_2 x_2 dx_2 + 2a_3 x_3 dx_3,$$

$$z = a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 + a_4;$$

and the general integral is expressed by the equations

$$z = a_1 x_1^2 + a_2 x_2^2 + \phi(a_1, a_2) x_3^2 + \psi(a_1, a_2),$$

$$0 = x_1^2 + \frac{d\phi}{da_1} x_3^2 + \frac{d\psi}{da_1},$$

$$0 = x_2^2 + \frac{d\phi}{da_2} x_3^2 + \frac{d\psi}{da_2},$$

where a_1, a_2 are functions of x_1, x_2, x_3 defined by the last two equations when the forms of ϕ, ψ are specified.

(18.) When the first integrals are not identical, a general integral *may*

be found in the same way as Boole has treated this case for two variables. But the results are very complex, and I believe that the best way to treat this case is to use a generalisation of Imschenetsky's method of variation of constants; viz., taking any particular solution involving $n+1$ constants, we find the most general relation between these constants, consistent with the given equation, is expressed by an equation of the second order, in which n constants are independent variables, and which is generally of an easily soluble form.

February 10th, 1876.

Prof. H. J. S. SMITH, F.R.S., President, in the Chair.

Messrs. Arthur Cockshott, M.A., late Fellow of Trinity College, Cambridge, Mathematical Master at Eton College, and Richard Thomas Wright, M.A., Fellow and Tutor of Christ's College, Cambridge, were proposed for election.

The following communications were made to the Society:—"Loci connected with the Rectangular Hyperbola, being inverses with respect to its centre and vertices:" Prof. Wolstenholme.—"On Determinants of Alternate Numbers:" W. Spottiswoode, F.R.S.—"On the Transformation of Gauss's Hypergeometric Series into a Continued Fraction:" Mr. T. Muir, M.A.—"On the Partition of Geometrical Curves:" the President.—"On the Sum of the Products of r different terms of a Series:" Mr. J. Hammond, B.A.—"On Pendular Motion:" Prof. Clifford, F.R.S.

A portion of a fly-sheet, by the Comte Léopold Hugo, Member of the Paris Mathematical Society, entitled "The Pan-imaginary Theory" (in which he speaks of space of $\frac{l}{m}$ dimensions), was read by the President. Messrs. Cayley, Clifford, Cotterill, Roberts, and the President took part in the discussions on the papers.

The following presents were made to the Society's Library:—

"The Journal of Education" (Jan., Feb., March, 1876), by Mr. Tucker, the Mathematical Editor.

"Proceedings of the Royal Society," Vol. xxiv., No. 166.

"Bulletin des Sciences," tome neuvième, Nov., Dec., 1875; and "Table des Matières et Noms d'Auteurs," tome viii., 1° sémestre, 1875.

"Mémoires de la Société des Sciences Physiques et Naturelles de

FIGURE 1.

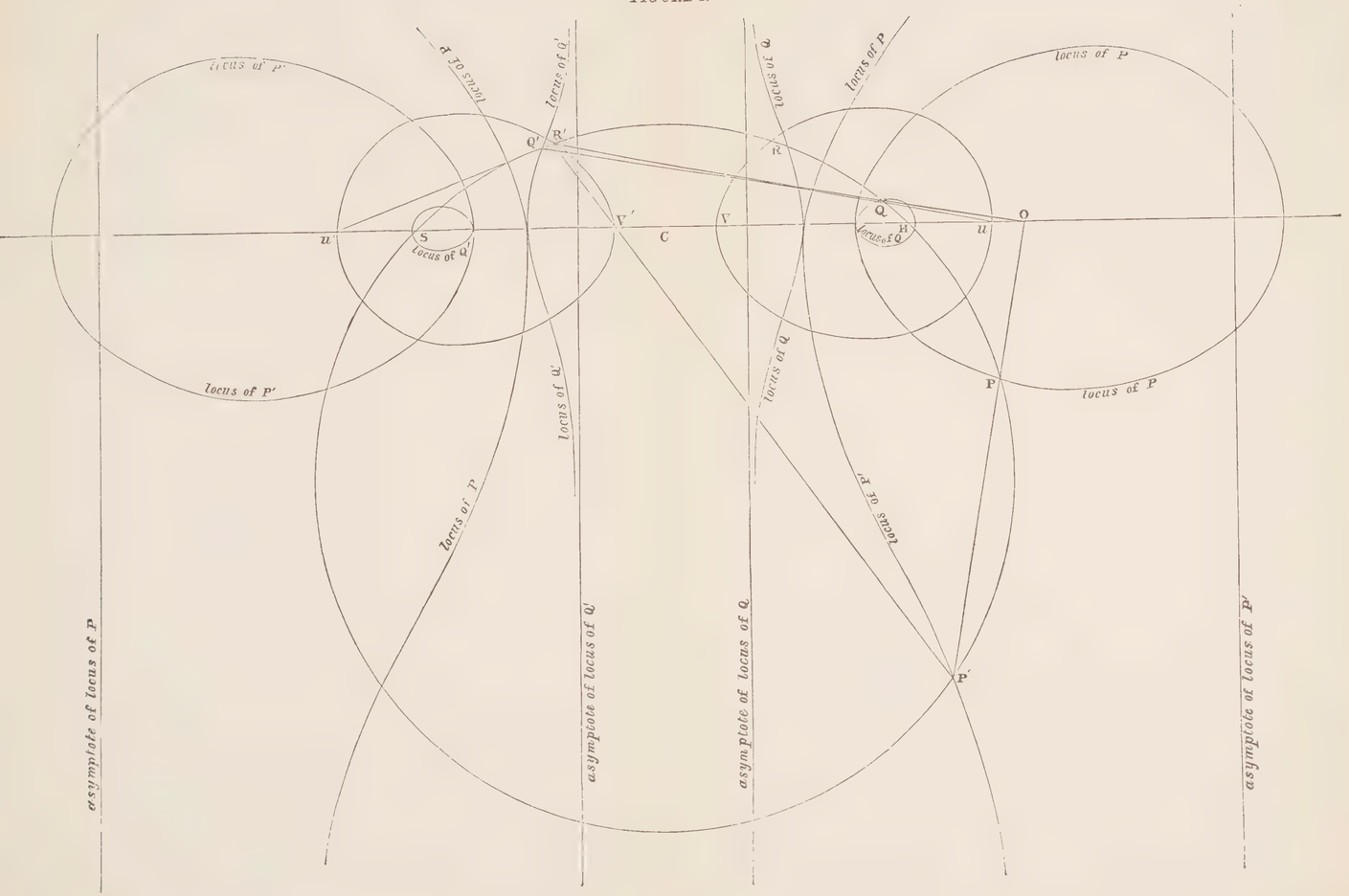
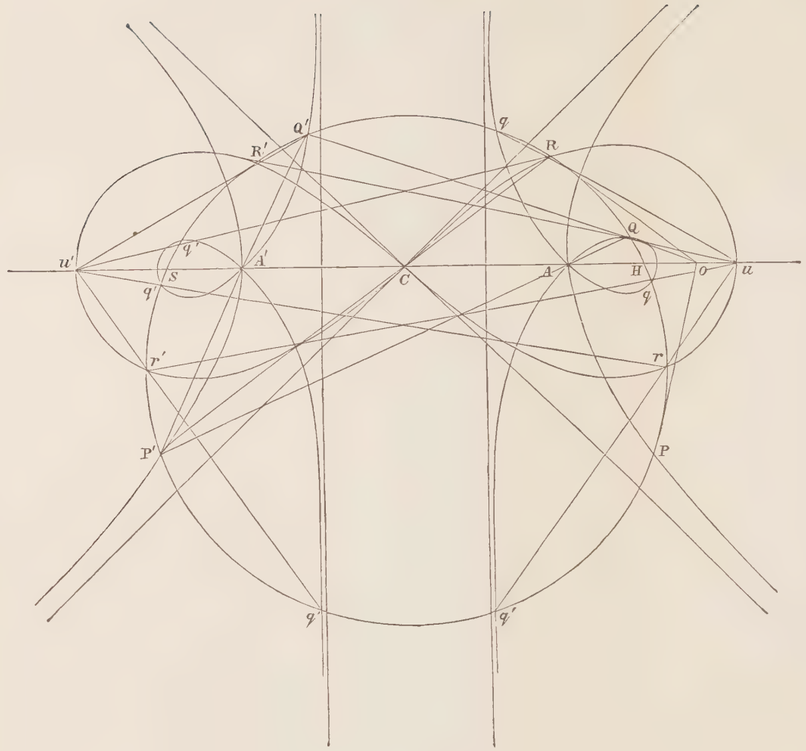


FIGURE 2.



Bordeaux," tome i. (2e série) 2e cahier, 1876; and "Extrait des Procès-verbaux des Séances," Bordeaux.

"Annual Report of the Board of Regents of the Smithsonian Institution," for the year 1874, Washington, 1875.

"Jornal de Sciencias, Mathematicas, Physicas, e Naturaes, poblicado sob os auspicios da Academia Real das Sciencias de Lisboa," tomo i., Nov. 1866—Dez. 1867, Lisboa, 1868; Tomo ii., Agosto 1868—Dez. 1869, Lisboa, 1870; Tomo iii., Junho 1870—Dez. 1871, Lisboa, 1871: Presented by W. S. W. Vaux, Esq.

On Theorems relating to the Circular Cubics which are the Inverses of a Lemniscate with respect to its Vertices. *By* Professor Wolstenholme, M.A.

[*Read February 10th*, 1876.]

1. Let S, H (Fig. 1) be two fixed points; R a variable point moving so that $SR.HR : SH^2 = 1 : 4n$; and let the circle SRH meet the lemniscate which is the locus of R again in R′ (on the same side of SH as R). Then let P, P′, Q, Q′ be points on this circle such that the distance of any one of them from R is a mean proportional between its distances from S, H; Q being always taken on the arc HR, Q′ on the arc RS, P, P′ on the arc of the circle opposite R, P being the one nearer to H in passing from H to S along this arc. This paper proposes to investigate the loci of these points and to prove various relations between their loci and the locus of R.

To determine the position of these points for a given position of R, refer the system to areal coordinates measured on the triangle RHS (these answering for the sake of familiar notation to A, B, C, so that for the present $HS=a$, $SR=b$, $RH=c$, where $a^2 = 4nbc$. Then, at Q, x, y will be positive and z negative, and we shall have, from well known properties of the circle,

$$x : y : z = \frac{a}{RQ} : \frac{b}{HQ} : \frac{-c}{SQ};$$

or, since $RQ^2 = SQ.HQ$,

$$\frac{x^2}{a^2} = \frac{-yz}{bc};$$

but at any point of the circle

$$a^2yz + b^2zx + c^2xy = 0,$$

whence, at Q,

$$-bcx^2 + b^2zx + c^2xy = 0, \quad \text{or} \quad x = \frac{b}{c}z + \frac{c}{b}y.$$

Similarly, at Q′,

$$x : y : z = \frac{a}{\mathrm{RQ}'} = \frac{-b}{\mathrm{HQ}'} = \frac{c}{\mathrm{SQ}'}, \quad \text{or, as before,} \quad \frac{x^2}{a^2} = \frac{-yz}{bc},$$

and we have the equation $x = \frac{b}{c}z + \frac{c}{b}y$

as that of the straight line QQ′.

At P, P′ we shall have y, z positive and x negative, so that $\frac{x^2}{a^2} = \frac{yz}{bc}$,

or the equation of PP′ will be

$$x + \frac{b}{c}z + \frac{c}{b}y = 0.$$

Since these equations can be written in the respective forms

$$x+y+z-(b+c)\left(\frac{y}{b}+\frac{z}{c}\right) = 0,$$

$$x+y+z-(b-c)\left(\frac{y}{b}-\frac{z}{c}\right) = 0,$$

we see that they are parallel to the external and internal bisectors of the angle HRS; and they also meet SH in the same point as the tangent at R to the circle. Thus we get a very simple construction for determining the four points.

2. At Q, Q′ we have the equation

$$a^2yz + (b^2z+c^2y)\left(\frac{b}{c}z + \frac{c}{b}y\right) = 0,$$

or $$4nb^2c^2yz + (b^2z+c^2y)^2 = 0,$$

so that $$\frac{c^2y}{b^2z} = -(2n+1) \pm \sqrt{4n(n+1)} \equiv -(\sqrt{n+1} \mp \sqrt{n})^2$$

$$\equiv -\frac{\sqrt{n+1} \mp \sqrt{n}}{\sqrt{n+1} \pm \sqrt{n}};$$

whence, at Q,

$$\frac{x}{2\sqrt{n}\,bc} = \frac{y}{b^2(\sqrt{n+1}+\sqrt{n})} = \frac{-z}{c^2(\sqrt{n+1}-\sqrt{n})};$$

and, at Q′,

$$\frac{x}{2\sqrt{n}\,bc} = \frac{-y}{b^2(\sqrt{n+1}-\sqrt{n})} = \frac{z}{c^2(\sqrt{n+1}+\sqrt{n})}.$$

Similarly, at P, P′, $$(b^2z+c^2y)^2 - 4nb^2c^2yz = 0,$$

or $$\frac{c^2y}{b^2z} = \frac{\sqrt{n} \pm \sqrt{n-1}}{\sqrt{n} \mp \sqrt{n-1}};$$

and at P (where the ratio $y : z$ is greater than at P′),

$$\frac{-x}{2\sqrt{n}\,bc} = \frac{y}{b^2(\sqrt{n}+\sqrt{n-1})} = \frac{z}{c^2(\sqrt{n}-\sqrt{n-1})};$$

at P', $\dfrac{-x}{2\sqrt{n}\,bc} = \dfrac{y}{b^2(\sqrt{n}-\sqrt{n-1})} = \dfrac{z}{c^2(\sqrt{n}+\sqrt{n-1})}$.

These can, of course, be deduced from Q, Q' by putting $-n$ for n.

3. We will now prove that the locus of any one of these points is an inverse of the locus of R, R'.

At R', $y+z=0$, and $-a^2y^2+xy(c^2-b^2)=0$, or

$$\frac{x}{a^2} = \frac{y}{c^2-b^2} = \frac{z}{b^2-c^2}.$$

At P, $\dfrac{-2\sqrt{n}}{a^2}x = \dfrac{y}{kb^2} = \dfrac{kz}{c^2}$, putting k for $\sqrt{n}+\sqrt{n-1}$; so that the line R'P is

$$\begin{vmatrix} a^2, & c^2-b^2, & b^2-c^2 \\ \dfrac{-a^2}{2\sqrt{n}}, & kb^2, & \dfrac{c^2}{k} \\ x, & y, & z \end{vmatrix} = 0;$$

and where this meets SH, $x=0$, and

$$-y\left(\frac{b^2-c^2}{2\sqrt{n}}+\frac{c^2}{k}\right)+z\left(kb^2+\frac{c^2-b^2}{2\sqrt{n}}\right)=0,$$

or $$-\frac{y}{2\sqrt{n}}\left\{c^2[2n-1-2\sqrt{n(n-1)}]+b^2\right\}$$
$$+\frac{z}{2\sqrt{n}}\left\{b^2[2n-1+2\sqrt{n(n-1)}]+c^2\right\}=0,$$

or $$\{-y(\sqrt{n}-\sqrt{n-1})^2+z\}\{c^2+b^2(\sqrt{n}+\sqrt{n-1})^2\}=0,$$

or, finally, $$\frac{y}{\sqrt{n}+\sqrt{n-1}} = \frac{z}{\sqrt{n}-\sqrt{n-1}} = \frac{1}{2\sqrt{n}}.$$

The distances of this point from S, H are then

$$\mathrm{SH}\,.\,\frac{\sqrt{n}+\sqrt{n-1}}{2\sqrt{n}}, \quad \mathrm{SH}\,.\,\frac{\sqrt{n}-\sqrt{n-1}}{2\sqrt{n}},$$

the rectangle under which is $\dfrac{\mathrm{SH}^2}{4n}$, so that the point is one of the inner vertices of the lemniscate (locus of R); the one nearer H, or that marked V in the figure. We have then R', V, P in one straight line, whence

$$\mathrm{R'V}\,.\,\mathrm{VP} = \mathrm{SV}\,.\,\mathrm{VH} = \frac{\mathrm{SH}^2}{4n}.$$

The locus of P is thus the inverse of the locus of R' (or R), the constant of inversion being $\dfrac{-\mathrm{SH}^2}{4n}$, corresponding points being on opposite sides of the centre of inversion.

Similarly, U, V, V′, U′ being the vertices of the lemniscate, we get in exactly the same way that the loci of P′, Q, Q′ are all inverses of the lemniscate, with respect to the vertices V′, U, U′; R′ being always the corresponding point, and the constant of inversion being numerically the same for all.

For the loci of Q, Q′ corresponding points are always on the same side of the centre of inversion.

The loci of P′, Q′ are thus images of the loci of P, Q (as is obvious beforehand) with respect to the straight line bisecting SH at right angles; *i. e.*, are their inverses with respect to a point at an infinite distance on SH, the constant of inversion being still $\infty S . \infty H$.

4. Any other pair of the four loci are inverses one of another; thus, if we draw the chord Qq parallel to SH and join Pq, we shall find that Pq meets SH in a fixed point; or, since q is a point on the locus of Q′, the loci of P, Q′ are inverses one of the other.

The equation of Qq is $\lambda x = y + z$, where

$$\frac{\lambda a^2}{2\sqrt{n}} = b^2(\sqrt{n+1} + \sqrt{n}) - c^2(\sqrt{n+1} - \sqrt{n});$$

also, at q, $$\frac{a^2}{x} + \frac{b^2}{y} + \frac{c^2}{z} = 0,$$

so that $$\frac{\lambda a^2}{y+z} + \frac{b^2}{y} + \frac{c^2}{z} = 0,$$

a quadratic in $\frac{y}{z}$, of which one root is $-\frac{b^2(\sqrt{n+1} + \sqrt{n})}{c^2(\sqrt{n+1} - \sqrt{n})}$, whence,

at q, $$\frac{y}{z} = -\frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n+1} + \sqrt{n}},$$

or $$\frac{y}{\sqrt{n+1} - \sqrt{n}} = \frac{-z}{\sqrt{n+1} + \sqrt{n}} = \frac{\lambda x}{-2\sqrt{n}}.$$

But, at P, $$\frac{-2\sqrt{n}}{a^2}x = \frac{y}{b^2(\sqrt{n} + \sqrt{n-1})} = \frac{z}{c^2(\sqrt{n} - \sqrt{n-1})},$$

so that the equation of Pq is

$$\begin{vmatrix} \frac{2\sqrt{n}}{\lambda}, & \sqrt{n} - \sqrt{n+1}, & \sqrt{n} + \sqrt{n+1} \\ \frac{-a^2}{2\sqrt{n}}, & b^2(\sqrt{n} + \sqrt{n-1}, & c^2(\sqrt{n} - \sqrt{n-1}) \\ x, & y, & z \end{vmatrix} = 0,$$

so that, where this meets SH, $x=0$, and

$$y\left\{\frac{-a^2}{2\sqrt{n}}(\sqrt{n}+\sqrt{n+1})-\frac{2\sqrt{n}}{\lambda}c^2(\sqrt{n}-\sqrt{n-1})\right\}$$
$$+z\left\{\frac{2\sqrt{n}}{\lambda}b^2(\sqrt{n}+\sqrt{n-1})+\frac{a^2}{2\sqrt{n}}(\sqrt{n}-\sqrt{n+1})\right\}=0;$$

or, substituting for λ,

$$-y\{[b^2(\sqrt{n+1}+\sqrt{n})-c^2(\sqrt{n+1}-\sqrt{n})](\sqrt{n}+\sqrt{n+1})$$
$$+z\{2\sqrt{n}(\sqrt{n}+\sqrt{n-1})b^2 \qquad +2\sqrt{n}c^2(\sqrt{n}-\sqrt{n-1})\}$$
$$+[b^2(\sqrt{n+1}+\sqrt{n})-c^2(\sqrt{n+1}-\sqrt{n})](\sqrt{n}-\sqrt{n+1})\}=0,$$

or
$$-y\{b^2(\sqrt{n+1}+\sqrt{n})^2+c^2(\sqrt{n}-\sqrt{n-1})^2\}$$
$$+z\{b^2(\sqrt{n}+\sqrt{n-1})^2+c^2(\sqrt{n+1}-\sqrt{n})^2\}=0;$$

or, finally,
$$y(\sqrt{n+1}+\sqrt{n})^2=z(\sqrt{n}+\sqrt{n-1})^2,$$

so that Pq divides SH in the fixed ratio

$$(\sqrt{n}+\sqrt{n-1})^2:(\sqrt{n+1}+\sqrt{n})^2.$$

Thus the loci of P, Q′ are inverses of each other with respect to a point which must be a common vertex of both curves, since both have points at infinity and SH is a common axis, and the constant of inversion is the rectangle under the distances of this point from S, H, so that corresponding points are always on opposite sides of the centre of inversion.

Similarly with any other pair of the four loci, or with the same locus taken twice ; they are inverses of each other with respect to a common vertex, and the constant of inversion (in the equation $rr'=C$) is always the rectangle under the distances of this vertex from S, H, sign of course being regarded. Each locus has one vertex which is peculiar to itself, and is its own inverse with respect to this vertex with the same rule for the constant.

5. Since these loci are all inverses of a bicircular quartic with respect to a vertex, it is obvious that they are all axial circular cubics. They have, however, another common property from which their form and properties could be very easily deduced. We will take the case of P, whose coordinates are

$$\frac{-\lambda a^2}{2\sqrt{n}},\quad \lambda b^2(\sqrt{n}+\sqrt{n-1}),\quad \lambda c^2(\sqrt{n}-\sqrt{n-1}),$$

λ being given by the equation

$$\frac{1}{\lambda}=\frac{-a^2}{2\sqrt{n}}+b^2(\sqrt{n}+\sqrt{n-1})+c^2(\sqrt{n}-\sqrt{n-1}).$$

The coordinates of V are

$$0, \quad \frac{\sqrt{n}+\sqrt{n-1}}{2\sqrt{n}}, \quad \frac{\sqrt{n}-\sqrt{n-1}}{2\sqrt{n}};$$

whence

$$\begin{aligned} -\mathrm{PV}^2 = \quad & a^2\left(\lambda b^2 - \frac{1}{2\sqrt{n}}\right)\left(\lambda c^2 - \frac{1}{2\sqrt{n}}\right) \\ & + b^2\left(\frac{-\lambda a^2}{2\sqrt{n}}\right)\left(\lambda c^2 - \frac{1}{2\sqrt{n}}\right)(\sqrt{n}-\sqrt{n-1}) \\ & + c^2\left(\frac{-\lambda a^2}{2\sqrt{n}}\right)\left(\lambda b^2 - \frac{1}{2\sqrt{n}}\right)(\sqrt{n}+\sqrt{n-1}) \end{aligned}$$

$$= \lambda a^2 b^2\left(-\frac{1}{2\sqrt{n}} + \frac{\sqrt{n}-\sqrt{n-1}}{4n}\right) + \lambda a^2 c^2\left(-\frac{1}{2\sqrt{n}} + \frac{\sqrt{n}+\sqrt{n-1}}{4n}\right) + \frac{a^2}{4n}$$

$$= \lambda\frac{a^2}{4n}\left(-b^2(\sqrt{n}+\sqrt{n-1}) - c^2(\sqrt{n}-\sqrt{n-1}) + \frac{1}{\lambda}\right) = \frac{-\lambda a^4}{8n\sqrt{n}},$$

or

$$\mathrm{PV}^2 = \frac{\lambda a^4}{8n\sqrt{n}}.$$

So

$$-\mathrm{SP}^2 = a^2 y(z-1) + b^2(z-1)x + c^2 xy$$

(if x, y, z be coordinates of P)

$$= -a^2 y - b^2 x, \quad \text{since} \quad a^2 yz + \ldots + \ldots = 0,$$

or

$$\mathrm{SP}^2 = \lambda a^2 b^2\left(\sqrt{n}+\sqrt{n-1} - \frac{1}{2\sqrt{n}}\right) = \frac{\lambda a^2 b^2}{2\sqrt{n}}(\sqrt{n}+\sqrt{n-1})^2,$$

and

$$\mathrm{HP}^2 = \frac{\lambda a^2 c^2}{2\sqrt{n}}(\sqrt{n}-\sqrt{n-1})^2.$$

Hence

$$\mathrm{SP}\,.\,\mathrm{HP} = \frac{\lambda a^2 bc}{2\sqrt{n}} = \frac{\lambda a^4}{8n\sqrt{n}} = \mathrm{PV}^2.$$

In the same way,

$$\mathrm{P'V'}^2 = \mathrm{P'S}\,.\,\mathrm{P'H}, \quad \mathrm{QU}^2 = \mathrm{QS}\,.\,\mathrm{QH}, \quad \mathrm{Q'U'}^2 = \mathrm{Q'S}\,.\,\mathrm{Q'H}.$$

Since, by the original definition of the points $\mathrm{PR}^2 = \mathrm{PS}\,.\,\mathrm{PH}$, &c., we see that V, R are equidistant from P; U, R from Q, &c., supplying another simple way of determining the points when the vertices of the lemniscate have been found.

6. These four loci are all similar curves, the common vertex of any two being of course the centre of similarity for those two. This does not at first sight appear probable, since, although they are all generated according to the law $\rho^2 = r_1 r_2$, where ρ, r_1, r_2 are the distances from three points in a straight line, yet for P the point from which ρ is measured lies between the other two, and for Q beyond the line joining them. It turns out, however, that any such curve has a double set of

three such points, either set satisfying the law. Thus, if A, B, C be three points in order on a straight line, and a point P move so that $PA^2 = PB \cdot PC$, we can find another set A′, B′, C′ on the same straight line so that $PA'^2 = PB' \cdot PC'$. A, A′ will be found to divide B, C harmonically; and if AB′, AC′ be denoted by x, y, and AB, AC by b, c, we shall also find $\dfrac{x+y}{a+b} = \left(\dfrac{a-b}{a+b}\right)^2 = \dfrac{xy}{ab}$.

Now $UH = \dfrac{a}{2\sqrt{n}}(\sqrt{n+1} - \sqrt{n})$, $US = \dfrac{a}{2\sqrt{n}}(\sqrt{n+1} + \sqrt{n})$,

whence, if u, F, F′ be the second system for the locus of Q,

$$Uu = \frac{a}{2\sqrt{n(n+1)}},$$

and UF, UF′ are the roots of the equation

$$x^2 \frac{n+1}{n} - a\sqrt{\frac{n+1}{n}}\, x + \frac{a^2}{4n} = 0,$$

or $$UF = \frac{a}{2}\frac{\sqrt{n} + \sqrt{n-1}}{\sqrt{n+1}}, \quad UF' = \frac{a}{2}\frac{\sqrt{n} - \sqrt{n-1}}{\sqrt{n+1}}.$$

We have then $$uF = \frac{a}{2\sqrt{n(n+1)}}[n-1 + \sqrt{n(n-1)}],$$

$$F'u = \frac{a}{2\sqrt{n(n+1)}}[\sqrt{n(n-1)} - n + 1];$$

whence $F'u : uF = \sqrt{n} - \sqrt{n-1} : \sqrt{n} + \sqrt{n-1} = HV : VS$.

Since, then, u divides F, F′ in the same ratio as V divides SH, the loci of Q, P will be similar, their linear dimensions being as FF′ : SH, or as $\sqrt{n-1} : \sqrt{n+1}$.

7. The points S, H will be foci of all the loci (from the equation $\rho^2 = r_1 r_2$); and for a like reason F, F′ will be foci of the locus of Q. Thus we get the four axial foci of this locus, F dividing SH in the ratio

$$\sqrt{n(n+1)} - \sqrt{n(n-1)} + 1 : \sqrt{n(n+1)} + \sqrt{n(n-1)} - 1,$$

and F′ in the ratio

$$\sqrt{n(n+1)} + \sqrt{n(n-1)} + 1 : \sqrt{n(n+1)} - \sqrt{n(n-1)} - 1.$$

The two other axial foci of the locus of P will be found to lie, (1) f, corresponding to S in the case of Q, will lie in HS produced, its distances from S, H being as

$$1 + \sqrt{n(n+1)} - \sqrt{n(n-1)} : 1 + \sqrt{n(n+1)} + \sqrt{n(n-1)};$$

(2) f', corresponding to H in the case of Q, will lie in SH produced, its distances from S, H being as

$$-1 + \sqrt{n(n+1)} + \sqrt{n(n-1)} : -1 + \sqrt{n(n+1)} - \sqrt{n(n-1)}.$$

If the distances of P from S, H, f, f' be denoted by r_1, r_2, r_3, r_4, it will be found that two of the equations connecting them are

$$(1)\quad (\sqrt{n+1}+\sqrt{n-1}-2\sqrt{n})\,r_1+(\sqrt{n-1}-\sqrt{n+1}+2\sqrt{n})\,r_2 = 2\sqrt{n-1}\,r_3,$$

$$(2)\quad (\sqrt{n+1}-\sqrt{n-1}+2\sqrt{n})\,r_1-(\sqrt{n+1}+\sqrt{n-1}+2\sqrt{n})\,r_2 = 2\sqrt{n-1}\,r_4;$$

from which may of course be deduced the two connecting $r_2 r_3 r_4$, or $r_3 r_4 r_1$.

8. The common vertex of the loci of P, Q divides SH in the ratio

$$2\sqrt{n}+\sqrt{n+1}+\sqrt{n-1} : 2\sqrt{n}-\sqrt{n+1}-\sqrt{n-1},$$

that of the loci of P, Q′ in the ratio

$$2\sqrt{n}-\sqrt{n+1}+\sqrt{n-1} : 2\sqrt{n}+\sqrt{n+1}-\sqrt{n-1};$$

the single vertex of the locus of P lies in SH produced, its distances from S, H being as $(\sqrt{n}+\sqrt{n-1})^2 : (\sqrt{n}-\sqrt{n-1})^2$, and the single vertex of the locus of Q also in SH produced, its distances from S, H being as $(\sqrt{n+1}+\sqrt{n})^2 : (\sqrt{n+1}-\sqrt{n})^2$. Each curve consists of an oval and a sinuous branch; when R lies on that branch of the lemniscate around H, P, Q are on their respective ovals and P′, Q′ on their respective sinuous branches. Each curve has an asymptote at right angles to SH, that for the locus of P dividing HS in the ratio

$$1+2\sqrt{n(n-1)} : 1-2\sqrt{n(n-1)},$$

and that for the locus of Q dividing SH in the ratio

$$1+2\sqrt{n(n+1)} : 1-2\sqrt{n(n+1)}.$$

The figure has been drawn for the case of $n=\frac{25}{24}$, and if we take SH=70, we shall have SV=42, VH=28, SU=84, UH=−14; the common vertex of P, Q is at distances 63, 7 from S, H; that of P, Q′ at distances 14, 56; the single vertex of P lies in SH produced at a distance from H of 56; that of Q also in SH produced at a distance from H of 2. The asymptote of P meets HS produced at a distance 49 from S, and that of Q meets SH at distances from S, H of 47, 23.

9. It appears from the results of the preceding articles that the axial foci of any one of the four loci (other than S and H) will only be real when $n>1$, as also the vertices V, V′ of the lemniscate. A special case arises when $n=1$, in which case it appears that P, P′ coincide, as also V, V′ in the centre C.

In this case we shall have for the locus of P,

$$CP^2 = SP \,.\, HP,$$

or the locus is a rectangular hyperbola whose foci are S, H.

The two supplementary foci for any one of the loci coincide, but for P, P′ both go to infinity, and we must consider the line at infinity as part of the locus in order to make all the properties connecting the loci hold in this case (*e. g.*, that of similarity). The two coincident foci of the locus of Q coincide at the vertex of the hyperbola nearer to H, and the equation connecting its three focal distances is

$$(\sqrt{2}-1)\,r_1-(\sqrt{2}+1)\,r_2=\pm 2r_3,$$

the upper sign for the oval.

This locus is, of course, still the inverse of the rectangular hyperbola with respect to the vertex nearer H, the constant being $-CS^2$, and also the inverse of the lemniscate (Bernoulli's in this case) with respect to its vertex U nearer H, the constant being CS^2.

This curve is a well-known circular cubic (although I am ignorant of a name for it), its equation referred to its double focus as origin being

$$y^2=x^2\frac{c-x}{c+x},$$

and referred to its vertex $r=c\,(\sec\theta-\tan\theta)$.

There are some curious results in the separate investigation of the relations between the loci in this case, which was what originally attracted my attention; but I think perhaps it would be *de luxe* to give the work when all the results follow from the preceding. I will content myself with a general summary of the connecting properties in this case, referring to Fig. 2.

10. A circle is drawn through the foci S, H of a rectangular hyperbola, meeting it in P, P′ (on the same side of the axis; PO is a tangent to the circle at P meeting the axis of the hyperbola in O; OR is the other tangent to the circle from O, and OQQ′ is drawn at right angles to OP to meet the circle in Q, Q′; then are R, C, P′; Q, A, P′; Q′, A′, P′ respectively collinear, so that the loci of R, Q, Q′ are inverses of the hyperbola with respect to its centre and vertices, the constants being $-CS^2$, $-CA^2$. Moreover, if R′ be the point of the circle opposite (not contrary) to R, and therefore a point on the locus of R, Q′, R′, U′; U, Q, R′ are also respectively collinear, U, U′ being the vertices of the Bernoulli's lemniscate (foci S, H) which is the locus of R. Hence the loci of Q, Q′ are inverses of the locus of R with respect to its vertices, the constant being CS^2.

Also, $QR^2=QS\,.\,QH=QU^2$, $Q'R^2=Q'S\,.\,Q'H=Q'U'^2$.

If the lemniscate cut the circle again in r, r', Ur, Ur' will meet the circle again in points q on the locus of Q, and $U'r$, $U'r'$ in points q' on the locus of Q′; as also will UR, U′R. Of course the locus of Q or Q′ is its own inverse with respect to its vertex; and the two being images of each other are inverse with respect to a point ∞, the con-

stant being $\infty S . \infty H$. The points S, H are foci of all the loci, and the loci of Q, Q′ have double foci at A, A′ respectively; so that for the oval

$$(\sqrt{2}-1)\,SQ - (\sqrt{2}+1)\,HQ = 2AQ,$$

and for the infinite branch,

$$(\sqrt{2}+1)\,HQ - (\sqrt{2}-1)\,SQ = 2AQ.$$

Other properties perhaps worth mentioning are—(1) the straight line bisecting UR at right angles (which passes through Q, since QU = QR), and that bisecting U′R at right angles, intersect on the circle and on the conjugate axis of the hyperbola; (2) the angle Q′RP exceeds the angle PRQ by a right angle.

On Determinants of Alternate Numbers. *By* William Spottiswoode, M.A., F.R.S.

[*Read February 10th*, 1876.]

Some very interesting unpublished notes on determinants and other functions of alternate numbers, kindly communicated to me by Professor Clifford, were the occasion of the following investigation of the leading properties of such determinants. With his permission I have incorporated his theorems in the present paper, specifying in each case that which is due to him.

If $\lambda_1, \lambda_2, \ldots, \mu_1\mu_2, \ldots, \ldots$ be alternate numbers, their laws of combination are thus expressed:

$$\lambda_i^2 = 0, \ \ldots\ \lambda_1\lambda_2 + \lambda_2\lambda_1 = 0 \ \ldots\ \ldots\ \ldots\ (1).$$

Then, for determinants of alternate numbers, Hankel's fundamental theorem takes the following form:—If $\alpha_1, \alpha_2, \ldots$ be any other set of alternate numbers, then the continued product

$$(\alpha_1\lambda_1+\alpha_2\lambda_2+\ldots)\,(\alpha_1\mu_1+\alpha_2\mu_2+\ldots)\ \ldots = -\alpha_1\alpha_2\ldots \begin{matrix}\lambda_1, \lambda_2, \ldots \\ \mu_1, \mu_2\ldots \\ \ldots\quad\ldots\end{matrix} \quad (I.)$$

It will be convenient generally in this paper to adopt Sylvester's umbral notation, and whenever no ambiguity is likely to arise to drop the enclosing brackets; in accordance with which we shall generally write

$$\begin{matrix}\lambda, \mu, \ldots & \text{for} & \lambda_1, \lambda_2, \ldots \\ 1, 2, \ldots & ,, & \mu_1, \mu_2, \ldots \\ \ldots\ \ldots\ \ldots & & \end{matrix}$$

Now in the development of such a determinant, the factors entering

into each term may be arranged either in the natural order of the letters $\lambda, \mu, \ldots$, or in that of the suffixes 1, 2, ... But as these two arrangements give results differing in form, it is necessary to understand beforehand that the normal form of development will be that according to letters. Thus, for the degrees 2 and 3, we should have

$$\begin{aligned}
&\lambda, \mu \quad = \lambda_1\mu_2 - \lambda_2\mu_1 = \lambda_1\mu_2 + \mu_1\lambda_2, \\
&1, 2 \\
&\lambda, \mu, \nu = \lambda_1(\mu_2\nu_3 - \mu_3\nu_2) + \lambda_2(\mu_3\nu_1 - \mu_1\nu_3) + \lambda_3(\mu_1\nu_2 - \mu_2\nu_1) \\
&1, 2, 3 \\
&\qquad = \lambda_1(\mu_2\nu_3 + \nu_2\mu_3) + \mu_1(\nu_2\lambda_3 + \lambda_2\nu_3) + \nu_1(\lambda_2\mu_3 + \mu_2\lambda_3);
\end{aligned}$$

and generally, if on developing according to lines (letters, $\lambda, \mu, \ldots$) the terms have alternate signs on the rule of ordinary determinants; then, on developing according to columns (suffixes 1, 2, ...), the terms will all have the same sign. This result may be expressed by the following formulæ

$$\underset{1, 2, \ldots}{\lambda, \mu, \ldots} = \Sigma \pm \underset{1, 2}{\lambda, \mu \ldots} = \Sigma + \underset{\lambda, \mu}{1, 2, \ldots} \qquad \ldots\ldots(\text{II.}),$$

in which last two expressions it is to be understood that the order of symbols in the upper line is to be retained, and that in the lower permuted.

In the case of odd-degreed determinants, there is yet a third way in which the development may be arranged; *e.g.*,

$$\underset{1, 2, 3}{\lambda, \mu, \nu} = \lambda_1(\mu_2\nu_3 - \mu_3\nu_2) + \mu_1(\nu_2\lambda_3 - \nu_3\lambda_2) + \nu_1(\lambda_2\mu_3 - \lambda_3\mu_2),$$

in which neither the order of the letters, nor that of the suffixes, is retained. The possibility of this depends upon the fact that the signs of all the terms in the first stage of the development, whether according to lines or columns, are alike positive.

Beside the alternate numbers proper, there is a set of numbers complementary to them, or "concurrent numbers" as they may be called, whose laws of combination are expressed by the following formulæ,

$$\alpha'^2 = 0, \quad \beta'^2 = 0, \quad \ldots \quad \alpha'\beta' = \beta'\alpha', \quad \ldots$$

In fact, any even-degreed products of alternate numbers will form such a set, and in the same way that alternate numbers $\alpha_1, \alpha_2, \ldots$ serve to resolve a determinant $\underset{1, 2, \ldots}{\lambda, \mu, \ldots}$ into a line product

$$(\alpha_1\lambda_1 + \alpha_2\lambda_2 + \ldots)(\alpha_1\mu_1 + \alpha_2\mu_2 + \ldots)\ldots,$$

so do the complementary numbers $\alpha', \beta', \ldots$ serve to resolve it into a column product

$$(\alpha'\lambda_1 + \beta'\mu_1 + \ldots)(\alpha'\lambda_2 + \beta'\mu_2 + \ldots).$$

In fact,

$$(\alpha'\lambda_1+\beta'\mu_1)\ (\alpha'\lambda_2+\beta'\mu_2) = \alpha'\lambda_1\beta'\mu_2+\beta'\mu_1\alpha'\lambda_2 = \alpha'\beta'\ (\lambda_1\mu_2-\lambda_2\mu_1),$$

$$\begin{aligned}
&(\alpha'\lambda_1+\beta'\mu_1+\gamma'\nu_1)(\alpha'\lambda_2+\beta'\mu_2+\gamma'\nu_2)(\alpha'\lambda_3+\beta'\mu_3+\gamma'\nu_3)\\
&\quad = (\alpha'\lambda_1+\ldots)\ \{\ \beta'\mu_2\gamma'\nu_3+\gamma'\nu_2\beta'\mu_3\\
&\qquad\qquad +\gamma'\nu_2\alpha'\lambda_3+\alpha'\lambda_2\gamma'\nu_3\\
&\qquad\qquad +\alpha'\lambda_2\beta'\mu_3+\beta'\mu_2\alpha'\lambda_3\}\\
&\quad = (\alpha'\lambda_1+\ldots)\ \{\ \beta'\gamma'\ (\mu_2\nu_3+\mu_3\nu_2)\\
&\qquad\qquad +\gamma'\alpha'\ (\nu_2\lambda_3-\nu_3\lambda_2)\\
&\qquad\qquad +\alpha'\beta'\ (\lambda_2\mu_3-\lambda_3\mu_2)\}\\
&\quad = \alpha'\beta'\gamma'\ \{\lambda_1\ (\mu_2\nu_3-\mu_3\nu_2)+\mu_1\ (\nu_2\lambda_3-\nu_3\lambda_2)+\nu_1\ (\lambda_2\mu_3-\lambda_3\mu_2)\}\\
&\quad = \alpha'\beta'\gamma'\ \{\lambda_1\ (\mu_2\nu_3-\mu_3\nu_2)+\lambda_2\ (\mu_3\nu_1-\mu_1\nu_3)+\lambda_3\ (\mu_1\nu_2-\mu_2\nu_1)\}\\
&\quad = \alpha'\beta'\gamma'\ \begin{matrix}\lambda,&\mu,&\nu,\\1,&2,&3,\end{matrix}
\end{aligned}$$

and so on.

This being so, it will be found, as remarked by Professor Clifford, that an interchange of two consecutive columns will, but that one of two consecutive rows will not, change the sign of the determinant. This may be expressed thus—

$$\begin{matrix}\lambda,&\mu,\ldots\\1,&2,\ldots\end{matrix} = -\begin{matrix}\mu,&\lambda,\ldots\\1,&2,\ldots\end{matrix} \qquad \begin{matrix}\lambda,&\mu,\ldots\\2,&1,\ldots\end{matrix} = -\begin{matrix}\lambda,&\mu,\ldots\\1,&2,\ldots\end{matrix} \qquad \ldots\ldots(\text{III.})$$

As a corollary to (III.) we have, when two columns are identical, in other words, when two suffixes are the same,

$$\begin{matrix}\lambda,&\mu,&\nu,\ldots\\1,&1,&3,\ldots\end{matrix} = \begin{matrix}\lambda_1,&\lambda_1,&\lambda_2,\ldots\\\mu_1,&\mu_1,&\mu_2,\ldots\\&\ldots\ldots&\end{matrix} = 0 \quad \ldots\ldots\ldots\ldots\ (\text{III. 1});$$

also, when two lines are identical, say $\lambda = \mu$,

$$\begin{matrix}\lambda,&\lambda,&\nu,\ldots\\1,&2,&3,\ldots\end{matrix} = 2\lambda_1\lambda_2\begin{matrix}\nu,&\rho,\ldots\\3,&4\ldots\end{matrix} - 2\lambda_1\lambda_3\begin{matrix}\nu,&\rho,\ldots\\1,&4,\ldots\end{matrix} + \ldots \qquad \ldots(\text{III. 2}).$$

For the addition and subtraction of determinants we have the ordinary formulæ:

$$\begin{matrix}\lambda_1,&\lambda_2+\lambda_1,&\lambda_2\\\mu_1,&\mu_2\quad\nu_1,&\nu_2\end{matrix} = \lambda_1\mu_2-\lambda_2\mu_1+\lambda_1\nu_1-\lambda_2\nu_1 = \begin{matrix}\lambda_1,&\lambda_2,\\\mu_1+\nu_1,&\mu_2+\nu_2,\end{matrix}$$

$$\begin{matrix}\lambda_1,&\lambda_2+\lambda_1,&\lambda_3\\\mu_1,&\mu_2\quad\mu_1,&\mu_3\end{matrix} = \lambda_1\mu_2-\lambda_2\mu_1+\lambda_1\mu_3-\lambda_3\mu_1 = \begin{matrix}\lambda_1,&\lambda_2+\lambda_3\,;\\\mu_1,&\mu_2+\mu_3\end{matrix}$$

and, generally,

$$\left.\begin{matrix}\lambda,&\mu,&\ldots\ \rho,&\ldots\ +\lambda,&\mu,&\ldots\ \rho',&\ldots = \lambda,&\mu,&\ldots\ \rho+\rho',\ldots\\1,&2,&\ldots\ k,&\ldots\quad 1,&2,&\ldots\ k,&\ldots\quad 1,&2,&\ldots\ k,\ldots\\\lambda,&\mu,&\ldots\ \rho,&\ldots\ +\lambda,&\mu,&\ldots\ \rho,&\ldots = \lambda,&\mu,&\ldots\ \rho,\ldots\\1,&2,&\ldots\ k,&\ldots\quad 1,&2,&\ldots\ k',&\ldots\quad 1,&2,&\ldots\ k+k',\ldots\end{matrix}\right\} \ldots\ (\text{IV}).$$

For multiplication, consider in the first instance the elementary case

$$\begin{matrix}\lambda,&\mu\\1,&2\end{matrix}\ .\ \begin{matrix}\lambda',&\mu'\\1,&2\end{matrix} = \lambda_1\mu_2\lambda_1'\mu_2'-\lambda_1\mu_2\lambda_2'\mu_1'-\lambda_2\mu_1\lambda_1'\mu_2'+\lambda_2\mu\ \lambda^1\mu_1'.$$

On the other hand, if we develope the ordinary formula, viz.,

$$\lambda_1\lambda_1' + \lambda_2\lambda_2', \quad \lambda_1\mu_1' + \lambda_2\mu_2',$$
$$\mu_1\lambda_1' + \mu_2\lambda_2', \quad \mu_1\mu_1' + \mu_2\mu_2',$$

in the usual way, and then change the sign of the last term, we obtain

$$(\lambda_1\lambda_1'+\lambda_2\lambda_2')\,(\mu_1\mu_1'+\mu_2\mu_2') + (\lambda_1\mu_1'+\lambda_2\mu_2')\,(\mu_1\lambda_1'+\mu_2\lambda_2'),$$
$$\lambda_1\lambda_1'\mu_1\mu_1'+\lambda_1\lambda_1'\mu_2\mu_2'+\lambda_2\lambda_2'\mu_1\mu_1'+\lambda_2\lambda_2'\mu_2\mu_2' = -\underset{1,\,2}{\lambda,\,\mu}\ \underset{1,\,2}{\lambda',\,\mu}$$
$$-\lambda_1\lambda_1'\mu_1\mu_1'-\lambda_1\lambda_2'\mu_2\mu_1'-\lambda_2\lambda_1'\mu_1\mu_2'-\lambda_2\lambda_2'\mu_2\mu_2.$$

In fact, if we write for brevity

$$\lambda_1\lambda_1' + \ldots = (\lambda\lambda'), \quad \lambda_1\mu_1' + \ldots = (\lambda\mu'), \ldots,$$

we should have

$$-\underset{1,\,2}{\lambda,\,\mu}\ .\ \underset{1,\,2}{\lambda',\,\mu'} = (\lambda\lambda')(\mu\mu') + (\lambda\mu')(\mu\lambda'),$$

and the change of sign in the last term might have been provided for by writing the formula thus, $(\lambda\lambda')\,(\mu\mu') - (\lambda\mu')\,(\lambda'\mu)$; but this hardly expresses the meaning of the change of sign in a manner applicable to all cases. The reason of it is this: in the original product the terms are all understood to be arranged in the order $\lambda\mu\lambda'\mu'$ (with their proper suffixes); the term $(\lambda\lambda')(\mu\mu')$ must therefore undergo a change of sign due to the transposition of $\lambda'\mu$; while the term $(\lambda\mu')\ (\mu\lambda')$ will retain its sign. In other words, the two terms, instead of having different, must have the same sign.

This being so, the ordinary formula for the multiplication of determinants may be applied, viz.,

$$-\underset{1,\,2,\,\ldots}{\lambda,\,\mu,\,\ldots}\ \underset{1,\,2,\,\ldots}{\lambda',\,\mu',\,\ldots} = \left.\begin{matrix}(\lambda\lambda'),\ (\lambda\mu'),\ \ldots \\ (\mu\lambda'),\ (\mu\mu'),\ \ldots \\ \ldots\quad\ \ldots\quad\ \ldots\end{matrix}\right\} \ \ldots\ldots\ldots\ (\text{V.}),$$

if it be understood that, after developing the right-hand side of the equation according to the ordinary rule, those terms which require an odd number of changes to bring them into the form $\lambda\mu\ \ldots\ \lambda'\mu'\ \ldots$ are to have their signs altered, while those which require an even number of changes are to retain their signs.

When the determinants to be multiplied together become equal, in other words, when the power of a determinant is required, the question is much simplified; for then

$$(\lambda\lambda') = 0, \ \ldots, \ (\lambda\mu) + (\mu\lambda) = 0, \ \ldots;$$

and consequently, if we develope the expression

$$-\begin{pmatrix}\lambda,\ \mu,\ \ldots \\ 1,\ 2,\ \ldots\end{pmatrix}^2 = \left.\begin{matrix}0,\ (\lambda\mu), \ldots = \quad 0,\ (\lambda\mu), \ldots \\ (\lambda\mu),\ 0,\ \ldots \quad -(\lambda\mu), 0,\ \ldots \\ \ldots\quad\ \ldots\quad\ \ldots \qquad\quad \ldots\quad\ \ldots\quad\ \ldots\end{matrix}\right\} \ \ldots\ldots\ (\text{V. 1})$$

according to the usual rule, the terms will be already arranged in the form $\lambda\mu \ldots \lambda\mu \ldots$ as in the power originally proposed. The expression for the square of a determinant is in fact a skew symmetrical determinant; and the laws appertaining to such determinants in ordinary algebra will apply to those in alternate numbers.

One very important consequence of this is, that the powers of odd-degreed determinants of alternate numbers necessarily vanish.

Thus,

$$-\begin{pmatrix}\lambda, & \mu, & \nu\\ 1, & 2, & 3\end{pmatrix}^2 = \begin{matrix}0, & (\lambda\mu), & -(\nu\lambda)\\ -(\lambda\mu), & 0, & (\mu\nu)\\ (\nu\mu), & -(\mu\nu), & 0,\end{matrix} = 0 \ldots\ldots\ldots (\text{V. } 2).$$

Odd-degreed determinants of alternate numbers are in fact themselves alternate numbers.

For even degrees we have

$$-\begin{pmatrix}\lambda, & \mu\\ 1, & 2\end{pmatrix}^2 = -(\lambda\mu)^2 = -2\lambda_1\mu_1\lambda_2\mu_2 = 2\lambda_1\lambda_2\mu_1\mu_2,$$

$$-\begin{pmatrix}\lambda, & \mu, & \nu, & \rho\\ 1, & 2, & 3, & 4\end{pmatrix}^2 = \begin{matrix}0, & (\lambda\mu), & (\lambda\nu), & (\lambda\rho)\\ -(\lambda\mu), & 0, & (\mu\nu), & (\mu\rho)\\ -(\lambda\nu), & -(\mu\nu), & 0, & (\nu\rho)\\ -(\lambda\rho), & -(\mu\rho), & -(\nu\rho), & 0\end{matrix}$$

$$= (\mu\nu)^2(\lambda\rho)^2 + (\nu\lambda)^2(\mu\rho)^2 + (\lambda\mu)^2(\nu\rho)^2 + 2\{(\nu\lambda)(\lambda\mu)(\mu\rho)(\nu\rho) + (\lambda\mu)(\mu\nu)(\nu\rho)(\lambda\rho) + (\mu\nu)(\nu\lambda)(\lambda\rho)(\mu\rho)\}$$

$$= \{(\mu\nu)(\lambda\rho) + (\nu\lambda)(\mu\rho) + (\lambda\mu)(\nu\rho)\}^2.$$

Before proceeding further, I quote a remark by Professor Clifford: "Alternate numbers may be considered as given in *sets* of n at a time (like the coordinates of a point in n-fold space); and in that case it is convenient to regard the product of all the numbers in any set as equal to unity. Hence all the products of all but one of the numbers make a new set, the *reciprocal* numbers." In what follows this view will be adopted, viz., it will be supposed that $\lambda_1\lambda_2 \ldots \lambda_n = 1$, $\lambda_2\lambda_3 \ldots \lambda_n = \lambda_1^{-1}, \ldots$

Beside the method used above there is another, depending upon Hankel's fundamental theorem, which may be conveniently applied to the powers of determinants. If $\alpha_1, \alpha_2, \ldots, \beta_1, \beta_2, \ldots, \ldots$ be any new sets of alternate numbers, then

$$\begin{Bmatrix}\lambda, & \mu, & \ldots\\ 1, & 2, & \ldots\end{Bmatrix}^2 = (\alpha_1\lambda_1 + \alpha_2\lambda_2 + \ldots)(\alpha_1\mu_1 + \ldots) \ldots (\beta_1\lambda_1 + \ldots)(\beta_1\mu_1 + \ldots) \ldots$$

$$= (\alpha_1\lambda_1 + \ldots)(\beta_1\lambda_1 + \ldots)(\alpha_1\mu_1 + \ldots)(\beta_1\mu_1 + \ldots) \ldots$$

$$= (\lambda_1\lambda_2\,\alpha, \beta + \ldots)(\mu_1\mu_2\,\alpha, \beta + \ldots) \ldots$$
$$\qquad 1, 2 \qquad\qquad 1, 2$$

Similarly,

$$\left\{\begin{matrix}\lambda, \mu, \dots \\ 1, 2, \dots\end{matrix}\right\}^3 = (\lambda_1\lambda_2\lambda_3 \underset{1,2,3}{\alpha, \beta, \gamma} + \dots)\,(\mu_1\mu_2\mu_3 \underset{1,2,3}{\alpha, \beta, \gamma} + \dots)\dots,$$

and so on. Consequently

$$\left\{\begin{matrix}\lambda, \mu \\ 1, 2\end{matrix}\right\}^2 = \lambda_1\lambda_2\mu_1\mu_2 \left\{\begin{matrix}\alpha, \beta \\ 1, 2\end{matrix}\right\}^2 = -2\lambda_1\lambda_2\mu_1\mu_2 = -2,$$

$$\left\{\begin{matrix}\lambda, \mu, \nu \\ 1, 2, 3\end{matrix}\right\}^2 = (\lambda_2\lambda_3\underset{2,3}{\alpha, \beta} + \lambda_3\lambda_1\underset{3,1}{\alpha, \beta} + \lambda_1\lambda_2\underset{1,2}{\alpha, \beta})(\mu_2\mu_3\underset{2,3}{\alpha, \beta} + \dots)(\nu_2\nu_3\underset{2,3}{\alpha, \beta} + \dots)$$

Now, from the value of $\left\{\begin{matrix}\alpha, \beta \\ 2, 3\end{matrix}\right\}^2$ it will easily be seen, as will be more fully discussed below, that,

$$\left\{\begin{matrix}\alpha, \beta \\ 2, 3\end{matrix}\right\}^3 = 0, \quad \left\{\begin{matrix}\alpha, \beta \\ 2, 3\end{matrix}\right\}^2 \left\{\begin{matrix}\alpha, \beta \\ 3, 1\end{matrix}\right\} = 0;$$

also
$$\underset{2,3}{\alpha, \beta} \cdot \underset{3,1}{\alpha, \beta} \cdot \underset{1,2}{\alpha, \beta} = -\underset{2,3}{\alpha, \beta}\,\alpha_1\beta_1\,(\alpha_2\beta_3 + \alpha_3\beta_2)$$
$$= -\alpha_1\beta_1\,(\alpha_3\beta_2\,\alpha_2\beta_3 - \alpha_2\beta_3\,\alpha_3\beta_2) = 0;$$

and consequently, as found before,

$$\left\{\begin{matrix}\lambda, \mu, \nu \\ 1, 2, 3\end{matrix}\right\}^3 = 0, \quad \left\{\begin{matrix}\lambda, \mu, \nu \\ 1, 2, 3\end{matrix}\right\}^4 = 0, \ \dots\dots$$

A similar process will apply to the case of the fifth degree. But as the theorem has been already proved, I will merely indicate the principal steps, as an exemplification of the present process.

$$\left\{\begin{matrix}\lambda, \mu, \nu, \rho, \sigma \\ 1, 2, 3, 4, 5\end{matrix}\right\}^2 = (\alpha_1\lambda_1 + \dots)(\alpha_1\mu_1 + \dots)(\alpha_1\nu_1 + \dots)(\alpha_1\rho_1 + \dots)(\alpha_1\sigma_1 + \dots)$$
multiplied by similar factors in β

$$= (\lambda_2\lambda_3\underset{2,3}{\alpha, \beta} + \lambda_3\lambda_1\underset{3,1}{\alpha, \beta} + \lambda_1\lambda_2\underset{1,2}{\alpha, \beta}$$
$$+ \lambda_4\lambda_5\underset{4,5}{\alpha, \beta} + \lambda_5\lambda_3\underset{5,3}{\alpha, \beta} + \lambda_3\lambda_4\underset{3,4}{\alpha, \beta}$$
$$+ \lambda_1\lambda_4\underset{1,4}{\alpha, \beta} + \lambda_1\lambda_5\underset{1,5}{\alpha, \beta} + \lambda_2\lambda_4\underset{2,4}{\alpha, \beta} + \lambda_2\lambda_5\underset{2,5}{\alpha, \beta}),$$
multiplied by similar factors in μ, ν, ρ, σ

$= (A_\lambda + B_\lambda + C_\lambda)$ multiplied by similar factors in μ, ν, ρ, σ, suppose.

Dropping the suffixes for a moment, and writing A^2, A^3, ... for AA, AAA, ... with any suffixes whatever, it will be seen from the combinations of $\underset{2,3}{\alpha, \beta}$, $\underset{3,1}{\alpha, \beta}$, ... therein involved, that terms of the following forms must vanish, viz.,

$$A^3 = 0, \quad A^4 = 0, \quad A^5 = 0, \quad B^3 = 0, \quad B^4 = 0, \quad B^5 = 0.$$

The terms which remain will consequently be of the forms C^5, C^4A, C^4B, C^3A^2, C^3B^2, C^3AB. Now in C^3 it will be found that the only terms which do not vanish will have as their $\alpha_1\beta_1$ coefficients the following combinations, viz.,

$$\alpha_2\alpha_3\alpha_4\alpha_5\beta_2\beta_3\beta_4\beta_5,$$
$$\alpha_1\alpha_3\alpha_4\alpha_5\beta_1\beta_3\beta_4\beta_5,$$
$$\alpha_1\alpha_2\alpha_3\alpha_5\beta_1\beta_2\beta_3\beta_5,$$
$$\alpha_1\alpha_2\alpha_3\alpha_4\beta_1\beta_2\beta_3\beta_4;$$

and consequently terms of the following forms vanish, viz.,

$$C^5 = 0,\ C^4A = 0,\ C^4B = 0.$$

The same reasoning will show that the terms

$$C^3A^2 = 0,\ C^3B^2 = 0;$$

so that the only terms remaining are of the form C^3AB, and of these all obviously vanish excepting those which have as factors the following expressions (in which for brevity 12 is written for α, β $\begin{matrix}1,\ 2\end{matrix}$):

$$(14^2 . 25 + \tfrac{1}{2}\,15 . 24 . 25)\ 23 . 53 = (14^2 + \tfrac{1}{2}\,15 . 24)\ 25 . 23 . 53,$$
$$(15^2 . 24 + \tfrac{1}{2}\,24 . 25 . 14)\ 23 . 34 = (15^2 + \tfrac{1}{2}\,25 . 14)\ 24 . 23 . 34,$$
$$(24^2 . 15 + \tfrac{1}{2}\,25 . 14 . 15)\ 31 . 53 = (24^2 + \tfrac{1}{2}\,25 . 14)\ 15 . 31 . 53,$$
$$(25^2 . 14 + \tfrac{1}{2}\,14 . 15 . 24)\ 31 . 34 = (25^2 + \tfrac{1}{2}\,15 . 24)\ 14 . 31 . 34,$$

all of which vanish in virtue of the outside factor. Hence finally, as before, the square and higher powers of the determinant of the fifth degree vanish.

It is hardly necessary to go through the calculations for the square of the determinant of the fourth degree; but the cube of the same determinant gives an important result. Proceeding as before,

$$\left\{\begin{matrix}\lambda, \mu, \nu, \rho \\ 1, 2, 3, 4\end{matrix}\right\}^3 = (\lambda_2\lambda_3\lambda_4\,\underset{2,3,4}{\alpha, \beta, \gamma} + \ldots) \times \text{terms in } \mu, \nu, \rho.$$

Now, if we write

$$\underset{2,3,4}{\alpha, \beta, \gamma} = A_1, \quad \underset{3,4,1}{\alpha, \beta, \gamma} = A_2, \ \ldots,$$

then will $\quad A_1^2 = 0,\ A_2^2 = 0,\ \ldots\ A_1A_2 + A_2A_1 = 0,\ \ldots;$

in other words, $A_1, A_2, \ldots$ are alternate numbers; and consequently we have

$$\left\{\begin{matrix}\lambda, \mu, \nu, \rho^3 \\ 1, 2, 3, 4\end{matrix}\right\} = (A_1\lambda_1^{-1} + \ldots)\ (A_1\mu_1^{-1} + \ldots)\ (A_1\nu_1^{-1} + \ldots)\ (A_1\nu_1^{-1} + \ldots)$$
$$= \underset{1,\ 2,\ 3,\ 4}{\lambda^{-1}, \mu^{-1}, \nu^{-1}, \rho^{-1}};$$

in other words, the cube of the determinant of the fourth degree is equal to the determinant of the reciprocal numbers.

Replacing the reciprocals by the products to which they are severally equal, we have

$$\left\{\begin{matrix}\lambda, \mu, \nu, \rho \\ 1, 2, 3, 4\end{matrix}\right\}^4 = (\alpha_1\lambda_1 + \ldots)\,(\beta_1\lambda_2\lambda_3\lambda_4 + \ldots) \times \text{terms in } \mu, \nu, \rho.$$
$$= -(\alpha_1\beta_1 + \alpha_2\beta_2 + \ldots)^4$$
$$= -1 \cdot 2 \cdot 3 \cdot 4.$$

And the same process enables us to generalise these results. In fact,

$$\left\{\begin{matrix}\lambda, \mu, \ldots \\ 1, 2, \ldots\end{matrix}\right\}^{n-1} = \left\{\begin{matrix}\lambda_1^{-1}\,\alpha, \beta, \ldots + \ldots \\ 2, 3, \ldots\end{matrix}\right\} \times \text{terms in } \mu, \ldots;$$

and if the quantities $\lambda, \mu, \ldots$ are even in number, the quantities $\alpha, \beta, \ldots$ will be odd; so that the coefficients of $\lambda_1^{-1}, \ldots$ are themselves alternate numbers (in virtue of the properties of odd-degreed determinants), and the result is

$$\left\{\begin{matrix}\lambda, \mu, \ldots \\ 1, 2, \ldots\end{matrix}\right\}^{n-1} = \left\{\begin{matrix}\lambda^{-1}, \mu^{-1}, \ldots \\ 1, \quad 2, \quad \ldots\end{matrix}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(VI.)};$$

and similarly

$$\left\{\begin{matrix}\lambda, \mu, \ldots \\ 1, 2, \ldots\end{matrix}\right\}^{n} = -1 \cdot 2 \ldots n \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(VII.)}$$

These results are due to Professor Clifford.

Compound determinants of alternate numbers present some peculiarities which distinguish them from ordinary compound determinants. Beginning with the simplest case, viz. those formed from, λ, μ, ν, we

$$\begin{matrix}\nu, \lambda & \nu, \lambda & & \\ 3, 1 & 1, 2 & = -\nu_1, \nu_2, \nu_3, . & = -\lambda_1, \lambda_1, \lambda_2, \lambda_3 = \lambda_1, \lambda_1, \lambda_2, \lambda_3. \\ & & \lambda_1, \lambda_2, \lambda_3, . & \quad \nu_1, \nu_1, \nu_2, \nu_3 \quad \mu_1, \mu_1, \mu_2, \mu_3 \\ \lambda, \mu & \lambda, \mu & . \lambda_2, \lambda_3, \lambda_1 & \quad . \lambda_1, \lambda_2, \lambda_3 \quad . \lambda_1, \lambda_2, \lambda_3 \\ 3, 1 & 1, 2 & . \mu_2, \mu_3, \mu_1 & \quad . \mu_1, \mu_2, \mu_3 \quad . \nu_1, \nu_2, \nu_3\end{matrix}$$

These results may be expressed by the following formulæ:

$$= -\left\|\begin{matrix}\lambda, \nu \\ 1, .\end{matrix}\right\| \cdot \left\|\begin{matrix}\lambda, \nu, \lambda, \mu \\ 1, . \quad 2, 3\end{matrix}\right\| \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(VIII.)},$$

$$= \left\|\begin{matrix}\lambda, \mu \\ 1, .\end{matrix}\right\| \cdot \left\|\begin{matrix}\lambda, \mu, \lambda, \nu \\ 1, . \quad 2, 3\end{matrix}\right\|$$

if it be understood that these expressions are to be thus developed (*e. g.*, taking the second)

$$= \lambda_1\,\mu, \lambda, \nu + \mu_1\,\lambda, \lambda, \nu$$
$$\quad 1, 2, 3 \qquad 1, 2, 3$$
$$= \lambda_1\,\lambda, \mu, \nu + 2\mu_1\,(\nu_1 : \lambda_1 + \nu_2 : \lambda_2 + \nu_3 : \lambda_3).$$
$$\quad 1, 2, 3$$

If the λ's had been ordinary instead of alternate numbers, the second term of this expression would have vanished, and we should have had the ordinary formula of compound determinants.

Again, if we form the determinant of all the first minors of the given

determinant, we shall find by a similar process the following expression :

$$\begin{matrix}\mu,\nu & \mu,\nu & \mu,\nu \\ 2,3 & 3,1 & 1,2\end{matrix} = \mu_1,\mu_2,\mu_3 \;.\;.\;. \\ \nu_1,\nu_2,\nu_3 \;.\;.\;. \\ \begin{matrix}\nu,\lambda & \nu,\lambda & \nu,\lambda \\ 2,3 & 3,1 & 1,2\end{matrix} \quad .\;.\;.\;\nu_1,\nu_2,\nu_3 \\ .\;.\;.\;\lambda_1,\lambda_2,\lambda_3 \\ \begin{matrix}\lambda,\mu & \lambda,\mu & \lambda,\mu \\ 2,3 & 3,1 & 1,2\end{matrix} \quad \lambda_1,\lambda_2,\lambda_3,\lambda_1,\lambda_2,\lambda_3 \\ \mu_1,\mu_2,\mu_3,\mu_1,\mu_2,\mu_3$$

$$= \begin{Bmatrix}\lambda,\mu,\nu \\ 1,2,3\end{Bmatrix}^2 + \begin{matrix}\mu,\mu,\nu & \lambda,\lambda,\nu, \\ 1,2,3 & 1,2,3\end{matrix}$$

or, resuming the notation adopted above,

$$= \left|\left|\begin{matrix}\lambda,\mu,\mu,\nu \\ 1,\;.\;2,3\end{matrix}\right|\right| \cdot \left|\left|\begin{matrix}\lambda,\mu,\lambda,\nu \\ 1,\;.\;2,3\end{matrix}\right|\right|$$

The first term of this expression, being the square of an odd-degreed determinant, vanishes, as was proved above ; and the second

$$= 4\,(\nu_1 : \mu_1 + \nu_2 : \mu_2 + \nu_3 : \mu_3)\,(\nu_1 : \lambda_1 + \nu_2 : \lambda_2 + \nu_3 : \lambda_3)$$
$$= 4\;\begin{matrix}\lambda^{-1}, & \mu^{-1}, & \nu^{-1}. \\ 1, & 2, & 3\end{matrix}$$

Hence the determinant of all the first minors of a determinant of the third degree is equal to four times the determinant of the reciprocals of the original constituents.

Proceeding to the fourth degree, and for brevity representing the coefficients of $\lambda_1, \lambda_2, \ldots$ in the development of λ, μ, ν, ρ, by $[\lambda_1], [\lambda_2], \ldots$

$$\begin{matrix}1, & 2, & 3, & 4\end{matrix}$$

we shall find $\begin{matrix}[\nu], & [\rho]\\ 3, & 4\end{matrix} = \begin{matrix}\lambda_1 & \lambda_2 & \lambda_4 & . & . & \lambda_3 \\ \mu_1 & \mu_2 & \mu_4 & . & . & \mu_3 \\ \rho_1 & \rho_2 & \rho_4 & . & . & \rho_3 \\ . & . & \lambda_4 & \lambda_1 & \lambda_2 & \lambda_3 \\ . & . & \mu_4 & \mu_1 & \mu_2 & \mu_3 \\ . & . & \nu_4 & \nu_1 & \nu_2 & \nu_3\end{matrix} = -\begin{matrix}\lambda_1 & \lambda_2 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ \mu_1 & \mu_2 & \mu_1 & \mu_2 & \mu_3 & \mu_4 \\ \nu_1 & \nu_2 & \nu_1 & \nu_2 & \nu_3 & \nu_4 \\ . & . & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ . & . & \mu_1 & \mu_2 & \mu_3 & \mu_4 \\ . & . & \rho_1 & \rho_2 & \rho_3 & \rho_4\end{matrix}$

$$= -\left|\left|\begin{matrix}\lambda,\mu,\nu \\ 1,\;2\;.\end{matrix}\right|\right| \cdot \left|\left|\begin{matrix}\lambda,\mu,\nu,\lambda,\mu,\rho \\ 1,\;.\;.\;2,3,4\end{matrix}\right|\right|$$

And similarly, by permuting letters and suffixes, we should obtain expressions for all other compound determinants of the same order. Passing to the next stage, we should find, after certain permutations,

$$\begin{matrix}[\mu], & [\nu], & [\rho]\\ 2, & 3, & 4\end{matrix} = \begin{matrix}\lambda_1 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ \mu_1 & \mu_1 & \mu_2 & \mu_3 & \mu_4 & \mu_1 & \mu_2 & \mu_3 & \mu_4 \\ \nu_1 & \nu_1 & \nu_2 & \nu_3 & \nu_4 & \nu_1 & \nu_2 & \nu_3 & \nu_4 \\ . & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & . & . & . & . \\ . & \mu_1 & \mu_2 & \mu_3 & \mu_4 & . & . & . & . \\ . & \rho_1 & \rho_2 & \rho_3 & \rho_4 & . & . & . & . \\ . & . & . & . & . & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ . & . & . & . & . & \nu_1 & \nu_2 & \nu_3 & \nu_4 \\ . & . & . & . & . & \rho_1 & \rho_2 & \rho_3 & \rho_4\end{matrix}$$

$$= \left|\left|\begin{matrix}\lambda,\mu,\nu \\ 1,\;.\;.\end{matrix}\right|\right| \cdot \left|\left|\begin{matrix}\lambda,\mu,\nu,\lambda,\mu,\rho \\ 1,\;.\;.\;2,3,4\end{matrix}\right|\right| \cdot \left|\left|\begin{matrix}\lambda,\mu,\nu,\lambda,\mu,\rho \\ 1,\;.\;.\;2,3,4\end{matrix}\right|\right|$$

which is to be developed in the same way as before.

From this expression we may, as in the case of the third degree, deduce the expression for the determinant of all the first minors of the given determinant, viz.,

$$= -\left\|\begin{matrix}\lambda, \mu, \nu, \mu, \nu, \rho \\ 1, \ . \ . \ 2, 3, 4\end{matrix}\right\| \cdot \left\|\begin{matrix}\lambda, \mu, \nu, \nu, \lambda, \rho \\ 1, \ . \ . \ 2, 3, 4\end{matrix}\right\| \cdot \left\|\begin{matrix}\lambda, \mu, \nu, \lambda, \mu, \rho \\ 1, \ . \ . \ 2, 3, 4\end{matrix}\right\|$$

The sign of the expression will in all cases be determined by the number of permutations of columns necessary to bring the same product on the principal diagonal in this as in the proposed expression, say $[\lambda_1]\,[\mu_2]\,[\nu_3]\,[\rho_4]$.

It is, perhaps, unnecessary to pursue these formulæ, which are quite general, further, except perhaps as a final example to write down the system of expressions for the compound determinants formed from two, three, four, and all of the first minors of a determinant of the fifth degree, viz.,

$$-\left\|\begin{matrix}\lambda, \mu, \nu, \rho \\ 1, 2, 3, \ .\end{matrix}\right\| \cdot \left\|\begin{matrix}\lambda, \mu, \nu, \rho, & \lambda, \mu, \nu, \sigma \\ 1, \ . \ . \ . & 2, 3, 4, 5\end{matrix}\right\|$$

$$\left\|\begin{matrix}\lambda, \mu, \nu, \rho \\ 1, 2, \ . \ .\end{matrix}\right\| \cdot \left\|\begin{matrix}\lambda, \mu, \nu, \rho, & \lambda, \mu, \nu, \rho \\ 1, \ . \ . \ . & 2, 3, 4, 5\end{matrix}\right\| \cdot \left\|\begin{matrix}\lambda, \mu, \nu, \rho, & \lambda, \mu, \rho, \sigma \\ 1, \ . \ . \ . & 2, 3, 4, 5\end{matrix}\right\|$$

$$-\left\|\begin{matrix}\lambda, \mu, \nu, \rho \\ 1, \ . \ . \ .\end{matrix}\right\| \cdot \left\|\begin{matrix}\lambda, \mu, \nu, \rho, & \lambda, \mu, \nu, \sigma \\ 1, \ . \ . \ . & 2, 3, 4, 5\end{matrix}\right\| \cdot \left\|\begin{matrix}\lambda, \mu, \nu, \rho, & \lambda, \mu, \rho, \sigma \\ 1, \ . \ . \ . & 2, 3, 4, 5\end{matrix}\right\| \cdot \left\|\begin{matrix}\lambda, \mu, \nu, \rho, & \lambda, \nu, \rho, \sigma \\ 1, \ . \ . \ . & 2, 3, 4, 5\end{matrix}\right\|,$$

$$\left\|\begin{matrix}\lambda, \mu, \nu, \rho, & \mu, \nu, \rho, \sigma \\ 1, \ . \ . \ . & 2, 3, 4, 5\end{matrix}\right\| \cdot \left\|\begin{matrix}\lambda, \mu, \nu, \rho, & \lambda, \nu, \rho, \sigma \\ 1, \ . \ . \ . & 2, 3, 4, 5\end{matrix}\right\| \cdot \left\|\begin{matrix}\lambda, \mu, \nu, \rho, & \lambda, \mu, \rho, \sigma \\ 1, \ . \ . \ . & 2, 3, 4, 5\end{matrix}\right\| \cdot \left\|\begin{matrix}\lambda, \mu, \nu, \rho, & \lambda, \mu, \nu, \sigma \\ 1, \ . \ . \ . & 2, 3, 4, 5\end{matrix}\right\|$$

and it is to be borne in mind that, in developing these expressions, the squares and higher powers of $\begin{matrix}\lambda, \mu, \nu, \rho, \sigma \\ 1, 2, 3, 4, 5\end{matrix}$, being of an odd degree, will vanish.

In the case of even-degreed determinants the evaluation of the determinant of all the first minors admits of a special solution. For example, that of the fourth degree

$$\begin{matrix}\mu, \nu, \rho & \mu, \nu, \rho & \mu, \nu, \rho & \mu, \nu, \rho \\ 2, 3, 4 & 3, 4, 1 & 4, 1, 2 & 1\ 2\ 3 \\ \nu, \rho, \lambda & & & \\ 2, 3, 4 & & & \end{matrix}$$

$$\vdots$$

$$= \begin{pmatrix} a_1\, \mu, \nu, \rho + \ldots \\ 2, 3, 4 \end{pmatrix} \begin{pmatrix} a_1\, \nu, \rho, \lambda + \ldots \\ 2, 3, 4 \end{pmatrix} \begin{pmatrix} a_1\, \rho, \lambda, \mu + \ldots \\ 2, 3, 4 \end{pmatrix} \begin{pmatrix} a_1\, \lambda, \mu, \nu + \ldots \\ 2, 3, 4 \end{pmatrix}$$

$$= (a_2^{-1} a_3^{-1} a_4^{-1}\, \mu, \nu, \rho + \ldots) \times \text{similar terms in } \nu\rho\lambda \times \ldots$$
$$2, 3, 4$$

But since $\alpha_1^{-1} = \alpha_2\alpha_3\alpha_4$, ... it follows that α_1^{-1}, ... are alternate numbers, and consequently the first factor written above

$$= (\alpha_1^{-1}\mu_1 + \alpha_2^{-1}\mu_2 + \ldots)\ (\alpha_1^{-1}\nu_1 + \alpha_2^{-1}\nu_2 + \ldots)\ (\alpha_1^{-1}\rho_1 + \alpha_2^{-1}\rho_2 + \ldots),$$

and the whole expression

$$= (\alpha_1^{-1}\lambda_1 + \ldots)^3\ (\alpha_1^{-1}\mu_1 + \ldots)^3\ (\alpha_1^{-1}\nu_1 + \ldots)^3\ (\alpha_1^{-1}\rho_1 + \ldots)^3$$
$$= 6^4\,(\alpha_1\lambda_1^{-1} + \ldots)\ (\alpha_1\mu_1^{-1} + \ldots)\ (\alpha_1\nu_1^{-1} + \ldots)\ (\alpha_1\rho_1^{-1} + \ldots)$$
$$= 6^4\,\lambda^{-1}\mu^{-1}\nu^{-1}\rho^{-1},$$
$$\quad 1,\ 2,\ 3,\ 4$$

and this method is obviously general.

Determinants sometimes occur as factors of a product having one or more lines common to more than one. When that is the case the product presents some peculiarities. This was pointed out by Prof. Clifford, from whose notes the following is a quotation:

$$\left.\begin{aligned} &\text{“}(\lambda_1\mu_2 - \lambda_2\mu_1)\ (\mu_1\nu_2 - \mu_2\nu_1) = \lambda_1\mu_2\mu_1\nu_2 + \lambda_2\mu_1\mu_2\nu_1 \\ &\qquad\qquad\qquad\qquad\qquad\quad = \mu_1\mu_2\,(\lambda_2\nu_1 - \lambda_1\nu_2 \\ &\qquad\qquad\qquad\qquad\qquad\quad = -\,(\lambda_1\nu_2 - \lambda_2\nu_1) \end{aligned}\right\} \ldots\ldots\ (\text{X.})$$

Hence, generally,

$$\left.\begin{aligned} &\lambda, \mu\ .\ \mu, \nu\ \ldots\ \sigma, \tau\ (n\text{-factors}) = (-)^{n-1}\,\lambda, \tau \\ &1,\ 2 \quad 1,\ 2 \quad\ 1,\ 2 \qquad\qquad\qquad\qquad\quad 1,\ 2 \end{aligned}\right\} \ldots\ldots\ (\text{XI.})$$

An analogous theorem holds for determinants of the nth order [when each determinant contains only two sets of numbers, but with repetitions]; viz., if we denote a determinant with r rows of λ, and s rows of μ, by (λ^r, μ^s), where $r+s = n$, then

$$\left.\begin{aligned} &|\ \lambda^r, \mu^s\ |\ \ldots\ |\ \mu^r, \nu^s\ |\ \ldots\ |\ \sigma^r, \tau^s\ | = (-)^{s(n+r)}\ (\lambda^r, \tau^s) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = (-)^s\ (\lambda^r, \tau^s) \end{aligned}\right\} \ldots\ldots\ (\text{XII.}),$$

and since $n+r = 2r+s = s\ (\text{mod. } 2)$; and so $s\,(n+r) \equiv s^2 = s\ (\text{mod. } 2)$.”

The following is a theorem supplementary to that first quoted:

$$\left.\begin{aligned} &\mu, \nu\ .\ \nu, \lambda\ .\ \lambda, \mu = -\,2 \\ &1,\ 2 \quad 1,\ 2 \quad 1,\ 2 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (\text{XIII.})$$

This admits of extension to higher degrees, thus

$$\begin{aligned} &\lambda, \mu, \nu\ .\ \mu, \nu, \rho\ .\ \nu, \rho, \lambda\ .\ \rho, \lambda, \mu = \lambda, \mu, \nu\ .\ \rho_1\rho_2\rho_3\ .\ [\lambda], [\mu], [\nu] \\ &1,\ 2,\ 3 \quad 1,\ 2,\ 3 \quad 1,\ 2,\ 3 \quad 1,\ 2,\ 3 \qquad\quad 1,\ 2,\ 3 \qquad\qquad 1, \quad 2, \quad 3 \\ &= \lambda_1[\lambda_1] + \ldots,\ \ \lambda_1[\mu_1] + \ldots,\ \ \lambda_1[\nu_1] + \ldots = \lambda\,[\lambda]\ \lambda\,[\mu]\ \lambda\,[\nu] \\ &\quad\ \mu_1[\lambda_1] + \ldots,\ \ \mu_1[\mu_1] + \ldots,\ \ \mu_1[\nu_1] + \ldots \qquad\ \mu\,[\lambda] \\ &\quad\ \nu_1[\lambda_1] + \ldots,\ \ \nu_1[\mu_1] + \ldots,\ \ \nu_1[\nu_1] + \ldots \qquad\ \nu\,[\lambda]. \end{aligned}$$

But $\lambda\,[\lambda] = \mu\,[\mu] = \nu\,[\nu] = \lambda, \mu, \nu = \nabla$, and consequently the products of these expressions vanish. Hence the whole expression,
$\qquad\qquad\qquad\quad 1,\ 2,\ 3$

$$= \nabla\,(-\mu\,[\nu]\ .\ \nu\,[\mu] - \lambda\,[\mu]\ .\ \mu\,[\lambda] - \lambda\,[\nu]\ .\ \nu\,[\lambda])$$
$$+ \lambda\,[\mu]\ \mu\,[\nu]\ \nu\,[\lambda] + \lambda\,[\nu]\ \mu\,[\lambda]\ \nu\,[\mu].$$

But since ∇ is a factor of the whole expression, it follows that, if we put $\lambda\,[\lambda] = 0$, $\mu\,[\mu] = 0$, $\nu\,[\nu] = 0$, the whole must vanish; in other words, the last two terms must vanish, and the whole

$$= -\nabla\,(\mu\,[\nu]\,.\,\nu\,[\mu] + \lambda\,[\mu]\,.\,\mu\,[\lambda] + \lambda\,[\nu]\,.\,\nu\,[\lambda]).$$

But
$$-\mu\,[\nu] = 2\,(\lambda_1 : \mu_1 + \lambda_2 : \mu_2 + \lambda_3 : \mu_3)$$
$$-\nu\,[\mu] = 2\,(\lambda_1 : \nu_1 + \lambda_2 : \nu_2 + \lambda_3 : \nu_3),$$

and consequently

$$\mu\,[\nu]\,.\,\nu\,[\mu] = 4\lambda^{-1}, \mu^{-1}, \nu^{-1}.$$
$$1,\ 2,\ 3$$

Hence the whole expression,

$$= -4\,\lambda, \mu, \nu\ \ \lambda^{-1}, \mu^{-1}, \nu^{-1}$$
$$1, 2, 3\ \ 1,\ 2,\ 3$$
$$= -4\,(\alpha_1\lambda_1 + \ldots)\,(\alpha_1\mu_1 + \ldots)\,(\alpha_1\nu_1 + \ldots)\,(\beta_1\lambda_2\lambda_3 + \ldots)$$
$$(\beta\,\mu_2\mu_3 + \ldots)\ (\beta\,\nu_2\nu_3 + \ldots)$$
$$= -4\,(\alpha_1\lambda_1 + \ldots)\,(\beta_1\lambda_2\lambda_3 + \ldots)\ \ldots$$
$$= -4\,(\alpha_1\beta_1 + \ldots)\,\lambda_1\lambda_2\lambda_3\ \ldots$$
$$= -4\,(\alpha_1\beta_1 + \ldots)^3$$
$$= -4\,.\,3\,.\,2\,.\,1.$$

It may be observed that the formula

$$\lambda, \mu, \ldots \sigma\ \ \mu, \nu, \ldots \tau \ldots = \lambda, \mu, \ldots \sigma\ \ [\lambda]\,[\mu]\,\ldots\,[\sigma]$$
$$1, 2, \ldots n\ \ 1, 2, \ldots n \qquad\qquad 1, 2, \ldots \nu\ \ 1,\ 2,\ \ldots\ \ n$$

is perfectly general; and any conclusions at which we can arrive about the determinant formed from the first minors of a given determinant will assist the evaluation. If we carry the process a step further, viz.,

$$\begin{matrix} \lambda\,[\lambda], & \lambda\,[\mu] & \ldots \\ \mu\,[\lambda], & \mu\,[\mu] & \ldots \\ \ldots & \ldots & \ldots \end{matrix}$$

then, in the case of odd-degreed functions, we may put

$$\begin{matrix} 0 & \lambda\,[\mu] & \ldots \\ \mu\,[\lambda] & 0 & \ldots \\ \ldots & \ldots & \ldots \end{matrix} = 0;$$

and afterwards we need retain only the terms having $\lambda\,[\lambda]$, $\mu\,[\mu]$, ... respectively as factors.

Sometimes determinants occur as factors of a product, such that their n columns are severally cyclical arrangements of $n-1$ out of n columns. For instance, as noticed by Prof. Clifford,

$$\lambda, \mu\ \ \lambda, \mu = -\lambda_2\mu_2\,(\lambda_1\mu_3 + \lambda_3\mu_1),$$
$$1, 2\ \ 2, 3$$

and, as a consequence,

$$\lambda, \mu\ \ \lambda, \mu\ \ \lambda, \mu = 0.$$
$$2, 3\ \ 3, 1\ \ 1, 2$$

For the third degree we should have, writing for brevity, 12 instead of μ, ν

$$\begin{matrix} & & & & & & & & & & & & & 1, 2 \\ & & \lambda, & \mu, & \nu & \lambda, & \mu, & \nu & \lambda, & \mu, & \nu & \lambda, & \mu, & \nu \\ & & 2, & 3, & 4 & 3, & 4, & 1 & 4, & 1, & 2 & 1, & 2, & 3 \end{matrix}$$

$$= \begin{pmatrix} . & \lambda_2 34 + \lambda_3 42 + \lambda_4 23 \\ (\lambda_1 34 & . & + \lambda_3 41 + \lambda_4 13) \\ (\lambda_1 24 + \lambda_2 41 & . & + \lambda_4 12) \\ (\lambda_1 23 + \lambda_2 31 + \lambda_3 12 & . &) \end{pmatrix} = \begin{matrix} . & 34, & -24, & 23 \\ -34, & . & 14, & 31 \\ 24, & -14, & . & 12 \\ -23, & -31, & -12, & . \end{matrix} = -1 \,.\, 2 \,.\, 3 \,.\, 4$$

$$= (23 \,.\, 14 + 31 \,.\, 24 + 12 \,.\, 34)^2.$$

From this it is easily seen that such a product can always be reduced to a skew symmetrical determinant; and consequently that, when the number (n) of columns is odd, the product will vanish. When the number is even, the product will

$$= -1 \,.\, 2 \,\ldots\, n. \quad \ldots\ldots\ldots\

a basis the general theorem* (recently discovered by the author) of which this is a case, that is to say, the theorem in which the series concerned is any series with coefficients following no specified law.

Beginning with the series

$$\begin{aligned} &B_0+B_1x+B_2x^2+B_3x^3+\ldots, \quad \text{or} \quad f_0,\\ &C_0+C_1x+C_2x^2+C_3x^3+\ldots, \quad \text{or} \quad f_1, \end{aligned}$$

and subtracting C_0 times the first from B_0 times the second, and dividing the result by x, we have

$$\begin{vmatrix} B_0 & B_1 \\ C_0 & C_1 \end{vmatrix} + \begin{vmatrix} B_0 & B_2 \\ C_0 & C_2 \end{vmatrix} x + \begin{vmatrix} B_0 & B_3 \\ C_0 & C_3 \end{vmatrix} x^2+\ldots\ldots, \quad \text{or} \quad f_2.$$

Next, subtracting $\begin{vmatrix} B_0 & B_1 \\ C_0 & C_1 \end{vmatrix}$ times the second series from C_0 times this third series, and dividing by x, there results

$$\begin{vmatrix} B_0 & B_1 & B_2 \\ 0 & C_0 & C_1 \\ C_0 & C_1 & C_2 \end{vmatrix} + \begin{vmatrix} B_0 & B_1 & B_3 \\ 0 & C_0 & C_2 \\ C_0 & C_1 & C_3 \end{vmatrix} x + \begin{vmatrix} B_0 & B_1 & B_4 \\ 0 & C_0 & C_3 \\ C_0 & C_1 & C_4 \end{vmatrix} x^2+\ldots\ldots, \quad \text{or} \quad f_3.$$

Again, subtracting $\begin{vmatrix} B_0 & B_1 & B_2 \\ 0 & C_0 & C_1 \\ C_0 & C_1 & C_2 \end{vmatrix}$ times the third series from $\begin{vmatrix} B_0 & B_1 \\ C_0 & C_1 \end{vmatrix}$ times the fourth, and dividing by C_0x, we find

$$\begin{vmatrix} B_0 & B_1 & B_2 & B_3 \\ 0 & B_0 & B_1 & B_2 \\ 0 & C_0 & C_1 & C_2 \\ C_0 & C_1 & C_2 & C_3 \end{vmatrix} + \begin{vmatrix} B_0 & B_1 & B_2 & B_4 \\ 0 & B_0 & B_1 & B_3 \\ 0 & C_0 & C_1 & C_3 \\ C_0 & C_1 & C_2 & C_4 \end{vmatrix} x + \begin{vmatrix} B_0 & B_1 & B_2 & B_5 \\ 0 & B_0 & B_1 & B_4 \\ 0 & C_0 & C_1 & C_4 \\ C_0 & C_1 & C_2 & C_5 \end{vmatrix} x^2+\ldots, \text{ or } f_4;$$

for, performing the operations, except the division by C_0x, upon the second terms of the series, we have

$$\begin{vmatrix} B_0 & B_1 \\ C_0 & C_1 \end{vmatrix} \cdot \begin{vmatrix} B_0 & B_1 & B_3 \\ 0 & C_0 & C_2 \\ C_0 & C_1 & C_3 \end{vmatrix} x - \begin{vmatrix} B_0 & B_1 & B_2 \\ 0 & C_0 & C_1 \\ C_0 & C_1 & C_2 \end{vmatrix} \cdot \begin{vmatrix} B_0 & B_2 \\ C_0 & C_2 \end{vmatrix} x,$$

$$\text{which} = \begin{vmatrix} B_0 & B_1 & 0 & 0 & B_2 \\ C_0 & C_1 & 0 & 0 & C_2 \\ 0 & B_2 & B_0 & B_1 & B_3 \\ 0 & C_1 & 0 & C_0 & C_2 \\ 0 & C_2 & C_0 & C_1 & C_3 \end{vmatrix} x = \begin{vmatrix} B_0 & B_1 & 0 & B_0 & B_2 \\ C_0 & C_1 & 0 & C_0 & C_2 \\ 0 & B_2 & B_0 & B_1 & B_3 \\ 0 & C_1 & 0 & C_0 & C_2 \\ 0 & C_2 & C_0 & C_1 & C_3 \end{vmatrix} x = C_0 \begin{vmatrix} B_0 & B_1 & B_2 & B_3 \\ 0 & B_0 & B_1 & B_2 \\ 0 & C_0 & C_1 & C_2 \\ C_0 & C_1 & C_2 & C_3 \end{vmatrix}$$

and the performance of the same operations upon the other terms of

* *Vide* Transactions of Roy. Soc. of Edin., 1875-76.

the series differs from this only in there being higher suffixes to the letters of the last column of all the determinants, except of course those which are the multipliers.

Continuing this process of deriving functions, the divisor next time being $\begin{vmatrix} B_0 & B_1 \\ C_0 & C_1 \end{vmatrix} x$, we have

$$\begin{vmatrix} B_0 & B_1 & B_2 & B_3 & B_4 \\ 0 & B_0 & B_1 & B_2 & B_3 \\ 0 & 0 & C_0 & C_1 & C_2 \\ 0 & C_0 & C_1 & C_2 & C_3 \\ C_0 & C_1 & C_2 & C_3 & C_4 \end{vmatrix} + \begin{vmatrix} B_0 & B_1 & B_2 & B_3 & B_5 \\ 0 & B_0 & B_1 & B_2 & B_4 \\ 0 & 0 & C_0 & C_1 & C_3 \\ 0 & C_0 & C_1 & C_2 & C_4 \\ C_0 & C_1 & C_2 & C_3 & C_5 \end{vmatrix} x + \begin{vmatrix} B_0 & B_1 & B_2 & B_3 & B_6 \\ 0 & B_0 & B_1 & B_2 & B_5 \\ 0 & 0 & C_0 & C_1 & C_4 \\ 0 & C_0 & C_1 & C_2 & C_5 \\ C_0 & C_1 & C_2 & C_3 & C_6 \end{vmatrix} x^2 + \ldots, \text{ or } f_5,$$

and so on.

Hence, denoting the first terms of the series by θ_0, θ_1, θ_2, ... respectively, we have the following equations:—

$$\begin{aligned} \theta_0 f_1 - \theta_1 f_0 &= x f_2, \\ \theta_1 f_2 - \theta_2 f_1 &= x f_3, \\ \theta_2 f_3 - \theta_3 f_2 &= \theta_1 x f_4, \\ \theta_3 f_4 - \theta_4 f_3 &= \theta_2 x f_5, \\ \ldots \quad &\ldots \quad \ldots \\ \theta_{n-1} f_n - \theta_n f_{n-1} &= \theta_{n-2} x f_{n+1}; \end{aligned}$$

whence

$$\frac{f_0}{f_1} = \frac{\theta_0}{\theta_1} - \frac{\frac{1}{\theta_1} x}{\frac{f_1}{f_2}},$$

$$\frac{f_1}{f_2} = \frac{\theta_1}{\theta_2} - \frac{\frac{1}{\theta_2} x}{\frac{f_2}{f_3}},$$

$$\frac{f_2}{f_3} = \frac{\theta_2}{\theta_3} - \frac{\frac{\theta_1}{\theta_3} x}{\frac{f_3}{f_4}},$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\frac{f_{n-1}}{f_n} = \frac{\theta_{n-1}}{\theta_n} - \frac{\frac{\theta_{n-2}}{\theta_n} x}{\frac{f_n}{f_{n+1}}};$$

and therefore

$$\frac{f_1}{f_0} = \cfrac{1}{\frac{\theta_0}{\theta} - \cfrac{\frac{1}{\theta_1}x}{\frac{\theta_1}{\theta_2} - \cfrac{\frac{1}{\theta_2}x}{\frac{\theta_2}{\theta_3} - \cfrac{\frac{\theta_1}{\theta_3}x}{\frac{\theta_3}{\theta_4} - \ddots - \cfrac{\frac{\theta_{n-2}}{\theta_n}x}{\frac{f_n}{f_{n+1}}}}}}};$$

so that we have

$$\frac{C_0 + C_1x + C_2x^2 + \ldots}{B_0 + B_1x + B_2x^2 + \ldots} = \cfrac{\theta_1}{\theta_0 - \cfrac{\theta_2 x}{\theta_1 - \cfrac{\theta_3 x}{\theta_2 - \cfrac{\theta_1\theta_4 x}{\theta_3 - \ddots - \cfrac{\theta_{n-3}\theta_n x}{\theta^{n-1} - \cfrac{\theta_{n-2}x f_{n+1}}{f_n}}}}}} \qquad \ldots\ldots\ldots(\text{I.}),$$

where $\theta_0 = B_0$, $\theta_1 = C_0$, $\theta_2 = \begin{vmatrix} B_0 & B_1 \\ C_0 & C_1 \end{vmatrix}$, &c.

Putting $B_0 = 1$, $B_1 = B_2 = \ldots\ldots = 0$, we find that

$$\theta_1,\ \theta_2,\ \theta_3,\ \theta_4,\ \theta_5,\ \theta_6, \ldots\ldots$$

become

$$C_0,\ C_1,\ \begin{vmatrix} C_0 & C_1 \\ C_1 & C_2 \end{vmatrix},\ \begin{vmatrix} C_1 & C_2 \\ C_2 & C_3 \end{vmatrix},\ \begin{vmatrix} C_0 & C_1 & C_2 \\ C_1 & C_2 & C_3 \\ C_2 & C_3 & C_4 \end{vmatrix},\ \begin{vmatrix} C_1 & C_2 & C_3 \\ C_2 & C_3 & C_4 \\ C_3 & C_4 & C_5 \end{vmatrix},\ \ldots\ldots$$

and denoting these by $\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \ldots\ldots$,
we have

$$C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots = \cfrac{\gamma_0}{1 - \cfrac{\gamma_1 x}{\gamma_0 - \cfrac{\gamma_2 x}{\gamma_1 - \cfrac{\gamma_1\gamma_3 x}{\gamma_2 - \ddots - \cfrac{\gamma_{n-2}\gamma_{n+1}x}{\gamma_n - \ddots}}}}} \qquad \ldots\ldots\ldots(\text{II.})$$

Again, putting $C_0 = 1$, $C_1 = C_2 = \ldots\ldots = 0$ in (1), we find, on writing

$$\beta_3, \beta_4, \beta_5, \ldots\ldots \text{ for } \begin{vmatrix} B_1 & B_2 \\ B_2 & B_3 \end{vmatrix}, \begin{vmatrix} B_2 & B_3 \\ B_3 & B_4 \end{vmatrix}, \begin{vmatrix} B_1 & B_2 & B_3 \\ B_2 & B_3 & B_4 \\ B_3 & B_4 & B_5 \end{vmatrix}, \ldots\ldots$$

$$B_0+B_1x+B_2x^2+B_3x^3+\ldots = B_0+\cfrac{B_1x}{1-\cfrac{B_2x}{B_1-\cfrac{\beta_3x}{B_2-\cfrac{B_1\beta_4x}{\beta_3-\ddots-\cfrac{\beta_{n-2}\beta_{n+1}x}{\beta_n-\ddots}}}}} \quad \ldots\ldots\ldots\text{(III.)},$$

as is also easily seen from (II.).

These results involve many interesting particular examples not hitherto obtained, and some even of those which have been obtained it is more simple to consider as cases of (I.), (II.), or (III.), than to deal with them as at present is commonly done.

Gauss' proposition above referred to is

$$F(\alpha,\beta+1,\gamma+1,x)-F(\alpha,\beta,\gamma,x)=\frac{\alpha(\gamma-\beta)}{\gamma(\gamma+1)}F(\alpha+1,\beta+1,\gamma+2,x) \quad \ldots\ldots\ldots(1);$$

and the result thence deduced is

$$\frac{F(\alpha,\beta+1,\gamma+1,x)}{F(\alpha,\beta,\gamma,x)} = \cfrac{\gamma}{\gamma-\cfrac{\alpha(\gamma-\beta)x}{\gamma+1-\cfrac{(\beta+1)(\gamma+1-\alpha)x}{\gamma+2-\cfrac{(\alpha+1)(\gamma+1-\beta)x}{\gamma+3-\cfrac{(\beta+2)(\gamma+2-\alpha)x}{\gamma+4-\ddots}}}}} \quad \ldots\ldots\ldots(A).$$

With this process and result of Gauss the following may be profitably compared:—

Putting $F(\alpha,\beta,\gamma,x)=y,$

the differential equation

$$(x^2-x)\frac{d^2y}{dx^2}+\left\{(\alpha+\beta+1)x-\gamma\right\}\frac{dy}{dx}+\alpha\beta y=0$$

is easily established; and since

$$\frac{dy}{dx}=\frac{\alpha\beta}{\gamma}F(\alpha+1,\beta+1,\gamma+1,x),$$

and

$$\frac{d^2y}{dx^2}=\frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)}F(\alpha+2,\beta+2,\gamma+2,x),$$

it follows that

$$(\alpha+1)\,(\beta+1)\,(x^2-x)\,\mathrm{F}\,(\alpha+2, \beta+2, \gamma+2, x)$$
$$+\{(\alpha+\beta+1)\,x-\gamma\}\,\mathrm{F}\,(\alpha+1, \beta+1, \gamma+1, x)+\gamma\,(\gamma+1)\,\mathrm{F}\,(\alpha, \beta, \gamma, x)=0 \quad \ldots\ldots\ldots(2),$$

which, being like (1) a relation between three of the functions, we can make use of in the same way. The result is

$$\frac{\mathrm{F}\,(\alpha+1, \beta+1, \gamma+1, x)}{\mathrm{F}\,(\alpha, \beta, \gamma, x)} = \quad \ldots\ldots\ldots\ldots\ldots(\mathrm{B})$$

$$= \cfrac{\gamma}{\gamma-(\alpha+\beta+1)\,x+\cfrac{(\alpha+1)\,(\beta+1)\,(x-x^2)}{\gamma+1-(\alpha+\beta+3)\,x+\ddots}}$$

$$\ddots+\cfrac{(\alpha+n)\,(\beta+n)\,(x-x^2)}{(\gamma+n)-(\alpha+\beta+2n+1)\,x+\cfrac{(\alpha+n+1)\,(\beta+n+1)\,(x-x^2)}{(\gamma+n+1)\,\frac{\mathrm{F}_{n+1}}{\mathrm{F}_{n+2}}}},$$

where F_{n+1} stands for $\mathrm{F}\,(\alpha+n+1, \beta+n+1, \gamma+n+1, x)$.

The expressions transformed into continued fractions in (A) and (B) are not the same, but, putting $\beta=0$ in (A), we have

$$\mathrm{F}\,(\alpha, 1, \gamma+1, x) = \quad \ldots\ldots\ldots(a);$$

$$= \cfrac{\gamma}{\gamma-\cfrac{\alpha\gamma x}{\gamma+1-\cfrac{1\,.\,(\gamma+1-\alpha)\,x}{\gamma+2-\cfrac{(\alpha+1)\,(\gamma+1)\,x}{\gamma+3-\cfrac{2\,(\gamma+2-\alpha)\,x}{\gamma+4-\ddots}}}}}$$

and, putting $\beta=0$ in (B), and then changing $\alpha+1$ into α, we find the same expression

$$= \cfrac{\gamma}{\gamma-\alpha x+\cfrac{1\,.\,\alpha\,(x-x^2)}{\gamma+1-(\alpha+2)\,x+\cfrac{2\,(\alpha+1)\,(x-x^2)}{\gamma+2-(\alpha+4)\,x+\ddots}}} \quad \ldots\ldots\ldots(b).$$

It is well known that the right-hand member of (A) may be continued *ad infinitum*, the correction to be added to any partial denominator in order to make the two numbers identical being of no account in the limit. The same, however, cannot be said regarding (B), and an instructive case worth noticing is where $\beta=\gamma$.

[It having been inadvertently stated, in the author's absence from the meeting at which the foregoing was read, that no law of formation was apparent in the determinants made use of for the expression of the continued fraction found as the equivalent of

$$\frac{C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \ldots}{B_0 + B_1 x + B_2 x^2 + B_3 x^3 + \ldots},$$

the determinant of the n^{th} order, denoted by θ_n in (I.), is herewith given, viz.:

$$\begin{vmatrix} B_0 & B_1 & B_2 & B_3 & \ldots\ldots & B_{n-1} \\ 0 & B_0 & B_1 & B_2 & \ldots\ldots & B_{n-2} \\ 0 & 0 & B_0 & B_1 & \ldots\ldots & B_{n-3} \\ \ldots & \ldots & \ldots & \ldots & \ldots\ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots\ldots & \ldots \\ 0 & 0 & C_0 & C_1 & \ldots\ldots & C_{n-3} \\ 0 & C_0 & C_1 & C_2 & \ldots\ldots & C_{n-2} \\ C_0 & C_1 & C_2 & C_3 & \ldots\ldots & C_{n-1} \end{vmatrix}$$

where the number of rows of B's is the same as the number of rows of C's, or greater than it by one, according as n is even or odd.

It is also worth while to add another continued fraction for comparison and contrast with that given in (I.) above. On using the method of Undetermined Coefficients to find the expansion of

$$\frac{C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \ldots}{B_0 + B_1 x + B_2 x^2 + B_3 x^3 + \ldots}$$

in ascending powers of x, the coefficients sought arise naturally as determinants from the solution of the equations to which we are led. Transforming this expansion into a continued fraction by Euler's method, we have

$$\frac{C_0 + C_1 x + C_2 x^2 + \ldots}{B_0 + B_1 x + B_2 x^2 + \ldots} = \cfrac{\delta_1}{B_0 - \cfrac{B_0 \delta_2 x}{B_0 \delta_1 + \delta_2 x - \cfrac{B_0 \delta_1 \delta_3 x}{B_0 \delta_2 + \delta_3 x - \cfrac{B_0 \delta_2 \delta_4 x}{B_0 \delta_3 + \delta_4 x - \ldots}}}}$$

where δ_n is the determinant

$$\begin{vmatrix} B_0 & B_1 & B_2 & B_3 & \ldots\ldots & B_{n-1} \\ 0 & B_0 & B_1 & B_2 & \ldots\ldots & B_{n-2} \\ 0 & 0 & B_0 & B_1 & \ldots\ldots & B_{n-3} \\ \ldots & \ldots & \ldots & \ldots & \ldots\ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots\ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots\ldots & B_1 \\ C_0 & C_1 & C_2 & C_3 & \ldots\ldots & C_{n-1} \end{vmatrix}$$

22nd Sept., 1876.]

On the Sum of the Products of r different Terms of a Series.

By J. Hammond, B.A.

[*Read February 10th*, 1876.]

Let $a_0, a_1, a_2, \ldots$ form a series, and let $C_r(a_0, a_1, \ldots a_{n-1})$ denote the sum of the products of r different terms, taken from the first n terms of this series.

Then

$$C_r(a_0, a_1, \ldots a_{n-1}, a_n) = C_r(a_0, a_1, \ldots a_{n-1}) + a_n C_{r-1}(a_0, a_1, \ldots a_{n-1}).$$

Or, writing $\phi(n, r)$ instead of $C_r(a_0, a_1, \ldots a_{n-1})$,

$$\phi(n+1, r) = \phi(n, r) + a_n \phi(n, r-1).$$

This gives

$$\begin{aligned} \phi(n, r) &= \Sigma a_n \phi(n, r-1) \\ &= \Sigma a_n \Sigma a_n \phi(n, r-2) \\ &= \quad \ldots \quad \ldots \quad \ldots \end{aligned}$$

where Σ means summation with respect to n, and each Σ operates on everything that follows it.

Continuing this process, we obtain

$$\phi(n, r) = \Sigma a_n \Sigma a_n \ldots \Sigma a_n \phi(n, 1),$$

there being $r-1$ operations, each expressed by Σa_n, performed on $\phi(n, 1)$ in succession; Σa_n meaning that its subject of operation is to be multiplied by a_n and then summed with respect to n.

This result may be written

$$\phi(n, r) = (\Sigma a_n)^{r-1} \phi(n, 1).$$

But

$$\phi(n, 1) = a_0 + a_1 + a_2 + \ldots + a_{n-1} = \Sigma a_n.$$

Thus

$$\phi(n, r) = (\Sigma a_n)^r \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

$\phi(n, r)$ or $(\Sigma a_n)^r$ is not the same as $C_r(a_0, a_1, \ldots a_{n-1})$ except in the case where each of the r arbitrary constants, implied in the r summations, is properly determined.

Let these constants be $c_1, c_2, \ldots c_r$.

Then

$$C_1(a_0, a_1, \ldots a_{n-1}) = \Sigma a_n + c_1.$$

Multiplying by a_n and summing with respect to n,

$$C_2(a_0, a_1, \ldots a_{n-1}) = (\Sigma a_n)^2 + c_1 \Sigma a_n + c_2.$$

Proceeding in this manner, the final result is

$$C_r(a_0, a_1, \ldots a_{n-1}) = (\Sigma a_n)^r + c_1 (\Sigma a_n)^{r-1} + c_2 (\Sigma a_n)^{r-2} + \ldots + c_r \quad \ldots (2).$$

The constants may now be found from the conditions that

$$\begin{aligned} C_1(a_0, a_1, \ldots a_{n-1}) &= 0 \quad \text{when } n = 0, \\ C_2(a_0, a_1, \ldots a_{n-1}) &= 0 \quad \text{when } n = 0 \text{ or } 1, \\ \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ C_r(a_0, a_1, \ldots a_{n-1}) &= 0 \quad \text{when } n = 0, 1, 2, \ldots r-1. \end{aligned}$$

We have

$$\left.\begin{aligned}
C_1(a_0, a_1, \ldots a_{n-1}) &= \phi(n,1)+c_1 \\
C_2(a_0, a_1, \ldots a_{n-1}) &= \phi(n,2)+c_1\phi(n,1)+c_2 \\
C_3(a_0, a_1, \ldots a_{n-1}) &= \phi(n,3)+c_1\phi(n,2)+c_2\phi(n,1)+c_3 \\
&\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
C_r(a_0, a_1, \ldots a_{n-1}) &= \phi(n,r)+c_1\phi(n,r-1)+c_2\phi(n,r-2)+\ldots+c_r
\end{aligned}\right\} (3).$$

In each of the equations (3) put $n=0$ and solve for c_r.

Then

$$\begin{vmatrix} 1 & 0 & 0 & \ldots \\ \phi(0,1) & 1 & 0 & \ldots \\ \phi(0,2) & \phi(0,1) & 1 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix} = \begin{vmatrix} \phi(0,1) & 1 & 0 & \ldots \\ \phi(0,2) & \phi(0,1) & 1 & \ldots \\ \phi(0,3) & \phi(0,2) & \phi(0,1) & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix} (-)^r c_r$$

(r rows) (r rows)

or
$$c_r = (-)^r \begin{vmatrix} \phi(0,1) & 1 & 0 & \ldots \\ \phi(0,2) & \phi(0,1) & 1 & \ldots \\ \phi(0,3) & \phi(0,2) & \phi(0,1) & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix} \quad \ldots\ldots\ldots\ldots (4).$$

(r rows)

Again, in the first equation, put $n=0$; in the second, $n=1$; in the third, $n=2$, and so on; and solve for c_r.

Then, as before,

$$c_r = (-)^r \begin{vmatrix} \phi(0,1) & 1 & 0 & \ldots \\ \phi(1,2) & \phi(1,1) & 1 & \ldots \\ \phi(2,3) & \phi(2,2) & \phi(2,1) & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix} \quad \ldots\ldots\ldots\ldots (5).$$

(r rows)

From (4) and (5) we obtain the identity

$$\begin{vmatrix} \phi(0,1) & 1 & 0 & \ldots \\ \phi(0,2) & \phi(0,1) & 1 & \ldots \\ \phi(0,3) & \phi(0,2) & \phi(0,1) & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix} \equiv \begin{vmatrix} \phi(0,1) & 1 & 0 & \ldots \\ \phi(1,2) & \phi(1,1) & 1 & \ldots \\ \phi(2,3) & \phi(2,2) & \phi(2,1) & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix}$$

Since, in order that the left-hand members of (3) may vanish, in the first equation n must be put $=0$; in the second, 0 or 1; in the third, 0, 1, or 2, and so on; there are $\lfloor r$ equations to find the r quantities $c_1, c_2, \ldots c_r$, and therefore $\lfloor r - r$ identical relations between the other quantities.

If $C_r(a_0, a_1, \ldots a_{n-1})$ be expressed in the form $(\Sigma a_n)^r$, where each summation is performed between certain limits, these limits are, for the first summation, from 0 to n; for the second, from 0 to n or from 1 to n; for the third, from 0 to n, from 1 to n, or from 2 to n; and so on.

Thus, $C_r(a_0, a_1, \dots a_{n-1})$ may be found in $\lfloor r$ different ways by taking different limits of summation; but the results will be in every case identical.

The most convenient limits are from 0 to n for each summation. For example,

$$C_1(a_0, a_1, \dots a_{n-1}) = \sum_0^n a_n,$$

$$C_2(a_0, a_1, \dots a_{n-1}) = (\sum_0^n a_n)^2 \equiv \sum_1^n a_n \sum_0^n a_n,$$

$$C_3(a_0, a_1, \dots a_{n-1}) = (\sum_0^n a_n)^3 \equiv \sum_1^n a_n (\sum_0^n a_n)^2 \equiv \sum_2^n a_n (\sum_0^n a_n)^2$$

$$\equiv \sum_0^n a_n \sum_1^n a_n \sum_0^n a_n \equiv (\sum_1^n a_n)^2 \sum_0^n a_n \equiv \sum_2^n a_n \sum_1^n a_n \sum_0^n a_n.$$

Corresponding to the determination of c_r in (4), we have

$$C_r(a_0, a_1, \dots a_{n-1}) = (\sum_0^n a_n)^r \quad \dots\dots\dots\dots\dots\dots (6);$$

this is the most convenient form.

And, corresponding to the determination of c_r in (5),

$$C_r(a_0, a_1, \dots a_{n-1}) = \sum_{r-1}^n a_n \sum_{r-2}^n a_n \dots \sum_0^n a_n \quad \dots\dots\dots\dots (7).$$

In the case where $\phi(\infty, r) = 0$ for all values of r, or, what is the same thing, where $C_r(a_0, a_1, \dots a_\infty)$ tends to a finite limit for all values of r, we have, from (3),

$$\left.\begin{aligned} c_1 &= C_1(a_0, a_1, \dots a_\infty) \\ c_2 &= C_2(a_0, a_1, \dots a_\infty) \\ c_3 &= C_3(a_0, a_1, \dots a_\infty) \\ \dots\ &\ \dots\ \ \dots\ \ \dots\ \ \dots \\ c_r &= C_r(a_0, a_1, \dots a_\infty) \end{aligned}\right\} \quad \dots\dots\dots\dots\dots\dots (8).$$

Hence c_r is a constant with respect to n only; *i. e.*, it is not independent of r or of the form of $a_0, a_1, a_2, \dots$, as is also evident from either (4) or (5).

In the most simple case, when $a_n = a^n$,

$$\phi(n, 1) = \Sigma a_n = \frac{a^n}{a-1},$$

$$\phi(n, 2) = (\Sigma a^n)^2 = \Sigma \frac{a^{2n}}{a-1} = \frac{a^{2n}}{(a-1)(a^2-1)},$$

$$\dots\quad \dots\quad \dots\quad \dots\quad \dots\quad \dots\quad \dots$$

$$\phi(n, r) = (\Sigma a^n)^r = \frac{a^{rn}}{(a-1)(a^2-1)\dots(a^r-1)};$$

and, from (3),

$$C_r(1, a, a^2, \dots a^{n-1}) = \phi(n, r) + c_1\phi(n, r-1) + c_2\phi(n, r-2) + \dots + c_r$$
$$= \frac{a^{nr} + A_1 a^{n(r-1)} + A_2 a^{n(r-2)} + \dots + A_r}{(a-1)(a^2-1)\dots(a^r-1)}.$$

The constants $A_1, A_2, \dots A_r$ are found from the condition that $C_r(1, a, a^2, \dots a^{n-1}) = 0$ when $n = 0, 1, \dots r-1$, and the final result is

$$C_r(1, a, a^2, \dots a^{n-1}) = \frac{(a^n-1)(a^n-a)\dots(a^n-a^{r-1})}{(a-1)(a^2-1)\dots(a^r-1)} \quad \dots\dots (9).$$

The term independent of n is c_r.

Thus,
$$c_r = (-)^r \frac{a^{1+2+\dots+(r-1)}}{(a-1)(a^2-1)\dots(a^r-1)}$$
$$= (-)^r \frac{a^{\frac{r(r-1)}{2}}}{(a-1)(a^2-1)\dots(a^r-1)}.$$

To verify (4) and (5), put $r = 2$.

Then
$$c_2 = \frac{a}{(a-1)(a^2-1)};$$

and

$$\begin{vmatrix} \phi(0,1) & 1 \\ \phi(0,2) & \phi(0,1) \end{vmatrix} = \begin{vmatrix} \frac{1}{a-1} & 1 \\ \frac{1}{(a-1)(a^2-1)} & \frac{1}{a-1} \end{vmatrix}$$
$$= \frac{1}{(a-1)^2} - \frac{1}{(a-1)(a^2-1)} = \frac{a}{(a-1)(a^2-1)},$$

$$\begin{vmatrix} \phi(0,1) & 1 \\ \phi(1,2) & \phi(1,1) \end{vmatrix} = \begin{vmatrix} \frac{1}{a-1} & 1 \\ \frac{a^2}{(a-1)(a^2-1)} & \frac{a}{a-1} \end{vmatrix}$$
$$= \frac{a}{(a-1)^2} - \frac{a^2}{(a-1)(a^2-1)} = \frac{a}{(a-1)(a^2-1)}.$$

In (6), put $a_n = a^n$ and $r = 2$. Then

$$C_2(1, a, \dots a^{n-1})$$
$$= (\sum_0^n a^n)^2 = \sum_0^n a^n \frac{a^n-1}{a-1}$$
$$= \frac{a^{2n}}{(a-1)(a^2-1)} - \frac{a^n}{(a-1)^2} - \frac{1}{(a-1)(a^2-1)} + \frac{1}{(a-1)^2}$$
$$= \frac{a^{2n} - a^n(a+1) + a}{(a-1)(a^2-1)}$$
$$= \frac{(a^n-1)(a^n-a)}{(a-1)(a^2-1)}.$$

In (7), put $a_n = a^n$ and $r = 2$. Then

$$C_2(1, a, \dots a^{n-1}) = \sum_1^n a^n \sum_0^n a^n = \sum_1^n a^n \frac{a^n - 1}{a - 1}$$

$$= \frac{a^{2n}}{(a-1)(a^2-1)} - \frac{a^n}{(a-1)^2} - \frac{a^2}{(a-1)(a^2-1)} + \frac{a}{(a-1)^2}$$

$$= \frac{(a^n-1)(a^n-a)}{(a-1)(a^2-1)} \quad \text{as before.}$$

In (9), put $n = \infty$, and suppose a to be fractional. Then the limit of $C_r(1, a, a^2, \dots a^{n-1})$ when n is infinite is

$$c_r = (-)^r \frac{a^{\frac{r(r-1)}{2}}}{(a-1)(a^2-1)\dots(a^r-1)},$$

which agrees with (8).

In (9), put $a = e^{\theta\sqrt{-1}}$. Then

$$C_r(1, e^{\theta\sqrt{-1}}, e^{2\theta\sqrt{-1}}, \dots e^{(n-1)\theta\sqrt{-1}}) = \frac{\sin\frac{n\theta}{2}\sin\frac{n-1}{2}\theta \dots \sin\frac{n-r+1}{2}\theta}{\sin\frac{\theta}{2}\sin\theta\dots\sin\frac{r\theta}{2}} e^{\frac{n-1}{2}r\theta\sqrt{-1}}.$$

But

$$C_r(1, e^{\theta\sqrt{-1}}, e^{2\theta\sqrt{-1}}, \dots e^{(n-1)\theta\sqrt{-1}})$$

$$= S\{\cos(\alpha+\beta+\dots r \text{ terms})\theta + \sqrt{-1}\sin(\alpha+\beta+\dots r \text{ terms})\theta\},$$

the sum of all terms of the form

$$\cos(\alpha+\beta+\dots)\theta + \sqrt{-1}\sin(\alpha+\beta+\dots)\theta,$$

where $\alpha, \beta, \dots$ are any r different terms of the series $0, 1, 2, \dots n-1$. With this notation,

$$S\cos(\alpha+\beta+\dots)\theta = \frac{\sin\frac{n\theta}{2}\sin\frac{n-1}{2}\theta \dots \sin\frac{n-r+1}{2}\theta}{\sin\frac{\theta}{2}\sin\theta\dots\sin\frac{r\theta}{2}} \cos\frac{n-1}{2}r\theta \quad \dots\dots (10);$$

and

$$S\sin(\alpha+\beta+\dots)\theta = \frac{\sin\frac{n\theta}{2}\sin\frac{n-1}{2}\theta \dots \sin\frac{n-r+1}{2}\theta}{\sin\frac{\theta}{2}\sin\theta\dots\sin\frac{r\theta}{2}} \sin\frac{n-1}{2}r\theta \quad \dots\dots (11).$$

Let (n) denote a given rational and integral function of n, of the order m; and write $a_n = (n)\,a^n$.

Then

$$\phi(n,1) = \Sigma(n)\,a^n = -\frac{a^n}{1-a}\cdot\frac{1}{1-\frac{a}{1-a}\Delta}(n),$$

$$\phi(n,2) = [\Sigma(n)\,a^n]^2 = \frac{a^{2n}}{1-a\,.\,1-a^2}\cdot\frac{1}{1-\frac{a^2}{1-a^2}\Delta}(n)\cdot\frac{1}{1-\frac{a}{1-a}\Delta}(n),$$

$$\phi(n,3) = [\Sigma(n)\,a^n]^3 = -\frac{a^{3n}}{1-a\,.\,1-a^2\,.\,1-a^3}\cdot\frac{1}{1-\frac{a^3}{1-a^3}\Delta}(n)$$
$$\times\frac{1}{1-\frac{a^2}{1-a^2}\Delta}(n)\cdot\frac{1}{1-\frac{a}{1-a}\Delta}(n),$$

... ...

$$\phi(n,r) = \frac{(-)^r a^{rn}}{1-a\,.\,1-a^2\,.\,1-a^r}\cdot\frac{1}{1-\frac{a^r}{1-a^r}\Delta}(n)\cdot\frac{1}{1-\frac{a^{r-1}}{1-a^{r-1}}\Delta}(n)\ \ldots\ldots$$
$$\ldots\ldots\frac{1}{1-\frac{a}{1-a}\Delta}(n)\ \ldots\ldots(12);$$

where, in the expressions on the right hand, Δ refers to n [viz., $\Delta(n) = (n+1)-(n)$ as usual]; the fractional symbols represent series $\frac{1}{1-\frac{a}{1-a}\Delta} = 1+\frac{a}{1-a}\Delta+\frac{a^2}{(1-a)^2}\Delta^2+$ &c., and where each power of Δ operates upon everything that follows it.

For instance, in the second formula, if to simplify $(n) = \overline{n+1}$ so that $\Delta(n) = 1$, $\Delta^2(n) = 0$, then omitting the factor $\frac{a^{2n}}{1-a\,.\,1-a^2}$, the remaining factor on the right hand is

$$\left(1+\frac{a^2}{1-a^2}\Delta+\frac{a^4}{(1-a^2)^2}\Delta^2\right)\overline{n+1}\left(1+\frac{a}{1-a}\Delta\right)\overline{n+1},$$

that is,

$$= \left(1+\frac{a^2}{1-a^2}\Delta+\frac{a^4}{(1-a^2)^2}\Delta^2\right)\overline{n+1}\left(\overline{n+1}+\frac{a}{1-a}\right)$$

$$= \left(1+\frac{a^2}{1-a^2}\Delta+\frac{a^4}{(1-a^2)^2}\Delta^2\right)\left(\overline{n+1}^2+\overline{n+1}\,.\,\frac{a}{1-a}\right)$$

$$= \overline{n+1}^2+\overline{n+1}\,.\,\frac{a}{1-a} = \overline{n+1}^2+\overline{n+1}\,.\,\frac{a}{1-a}$$
$$+\frac{a^2}{1-a^2}\Delta\left(\overline{n+1}^2+\overline{n+1}\,.\,\frac{a}{1-a}\right) \qquad +\frac{a^2}{1-a^2}\left(2n+3+\frac{a}{1-a}\right)$$
$$+\frac{a^4}{(1-a^2)^2}\Delta^2\left(\overline{n+1}^2+\overline{n+1}\,.\,\frac{a}{1-a}\right) \qquad +\frac{a^4}{(1-a^2)^2}\,.\,2.$$

Referring to (3) and (4), we see that, in the general case,

$$C_r(a_0, a_1, \dots a_{n-1}) = \begin{vmatrix} \phi(n,r) & \phi(n,r-1) & \phi(n,r-2) & \dots & 1 \\ \phi(0,1) & 1 & 0 & \dots & 0 \\ \phi(0,2) & \phi(0,1) & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \phi(0,r) & \phi(0,r-1) & \phi(0,r-2) & \dots & 1 \end{vmatrix} \quad \dots(13).$$

When $a_n = \overline{n+1}\,.\,a^n$,

$$C_1(1, 2a, \dots na^{n-1}) = \begin{vmatrix} \phi(n,1) & 1 \\ \phi(0,1) & 1 \end{vmatrix}$$

$$C_2(1, 2a, \dots na^{n-1}) = \begin{vmatrix} \phi(n,2) & \phi(n,1) & 1 \\ \phi(0,1) & 1 & 0 \\ \phi(0,2) & \phi(0,1) & 1 \end{vmatrix}$$

where
$$\phi(n,1) = -\frac{a^n}{1-a}\left(\overline{n+1} + \frac{a}{1-a}\right),$$

$$\phi(n,2) = \frac{a^{2n}}{(1-a)(1-a^2)} \times \left\{\overline{n+1}^2 + \overline{n+1}\,.\,\frac{a}{1-a} + \frac{a^2}{1-a^2}\left(2n+3+\frac{a}{1-a}\right) + \frac{2a^4}{(1-a^2)^2}\right\}.$$

Hence
$$\phi(0,1) = -\frac{1}{1-a}\left(1+\frac{a}{1-a}\right) = -\frac{1}{(1-a)^2},$$

$$\phi(0,2) = \frac{1}{(1-a)(1-a^2)}\left\{1+\frac{a}{1-a}+\frac{a^2}{1-a^2}\left(3+\frac{a}{1-a}\right)+\frac{2a^4}{(1-a^2)^2}\right\}$$

$$= \frac{1}{(1-a)(1-a^2)}\left(1+\frac{a}{1-a}+\frac{2a^2}{1-a^2}\right)\left(1+\frac{a^2}{1-a^2}\right)$$

$$= \frac{1+a+2a^2}{(1-a)(1-a^2)^3}.$$

$$C_1(1, 2a, \dots na^{n-1}) = \frac{1}{(1-a)^2} - \frac{a^n}{1-a}\left(\overline{n+1} + \frac{a}{1-a}\right)$$

$$= \frac{1}{(1-a)^2}(1 - \overline{n+1}\,a^n + na^{n+1}) \quad \dots\dots\dots(14),$$

which is easily verified.

$$C_2(1, 2a, \dots na^{n-1}) = \{\phi(0,1)\}^2 - \phi(0,2) - \phi(0,1)\,.\,\phi(n,1) + \phi(n,2)$$

$$= \frac{1}{(1-a)^4} - \frac{1+a+2a^2}{(1-a)(1-a^2)^3} - \frac{a^n}{(1-a)^3}\left(\overline{n+1} + \frac{a}{1-a}\right)$$

$$+ \frac{a^{2n}}{(1-a)(1-a^2)} \times \left\{\overline{n+1}^2 + \overline{n+1}\,\frac{a}{1-a} + \frac{a^2}{1-a^2}\left(2n+3+\frac{a}{1-a}\right) + \frac{2a^4}{(1-a^2)^2}\right\}$$

$$= \frac{2a+a^2+a^3}{(1-a)(1-a^2)^3} - \frac{a^n}{(1-a)^4}(n+1-na)$$
$$+ \frac{a^{2n}}{(1-a)(1-a^2)}\left\{n^2+n\left(2+\frac{a}{1-a}+\frac{2a^2}{1-a^2}\right)+\frac{1+a+2a^2}{(1-a^2)^2}\right\}$$
$$= \frac{1}{(1-a)(1-a^2)^3}\left\{2a+a^2+a^3-\overline{n+1}\,.\,a^n-\overline{2n+3}\,.\,a^{n+1}-3a^{n+2}\right.$$
$$+\overline{2n-1}\,.\,a^{n+3}+na^{n+4}+\overline{n+1}^2.\,a^{2n}+\overline{n+1}\,.\,a^{2n+1}$$
$$\left.-\overline{2n^2+n-2}\,.\,a^{2n+2}-n\,.\,a^{2n+3}+n\,.\,\overline{n-1}\,.\,a^{2n+4}\right\};$$

e. g., when $n=2$, this is

$$2a = \frac{1}{(1-a)(1-a^2)^3}\left\{2a-2a^2-6a^3+6a^4+6a^5-6a^6-2a^7+2a^8\right\},$$

the truth of which is evident.

When n is infinite and a fractional,

$$C_2(1,\ 2a,\ \ldots\ na^{n-1}) = \frac{2a+a^2+a^3}{(1-a)(1-a^2)^3} \quad \ldots\ldots\ldots\ldots\ (15).$$

This result is true approximately whenever na^n is small enough to be neglected.

When $n=10$, $a=\frac{1}{10}$, the result obtained from (15) is

$$\cdot 241,620,824,554,538\ \ldots,$$

the correct value of $C_2(1,\ 2a,\ \ldots\ 10a^9)$ being

$$\cdot 241,620,823,030,380,9.$$

It may be noticed that the term independent of n in (14) is $\frac{1}{(1-a)^2} = 1+2a+3a^2+\ \ldots$ *ad inf.* by expansion; that the term independent of n in the expression for $C_2(1,\ 2a,\ \ldots\ na^{n-1})$ is the limit of that expression when a is fractional and n infinite; and generally, since we have, from (12),

$$\phi(n,\ r) = (a^r)^n\,\mathrm{F}(n),$$

where $\mathrm{F}(n)$ is a rational integral function of n, when a is fractional, $\phi(\infty,\ r) = 0$ for all values of r, and the limit of $C_r[(0),\ (1)\ a,\ \ldots\ (n-1)\ a^{n-1}] = c_r$, the term in its expression independent of n.

The general term of $\phi(n,\ r)$ obtained from (12) by expansion is

$$\frac{(-)^r\,a^{rn}}{(1-a)(1-a^2)\ \ldots\ (1-a^r)}\left(\frac{a^r}{1-a^r}\Delta\right)^{\kappa_r}(n)\left(\frac{a^{r-1}}{1-a^{r-1}}\Delta\right)^{\kappa_{r-1}}(n)\ \ldots\ldots$$
$$\ldots\ldots\left(\frac{a}{1-a}\Delta\right)\ (n)$$
$$= \frac{(-)^r\,a^{rn+\kappa_1+2\kappa_2+\ \ldots\ +r\kappa_r}}{(1-a)^{\kappa_1+1}(1-a^2)^{\kappa_2+1}\ \ldots\ (1-a^r)^{\kappa_r+1}}\,\Delta^{\kappa_r}(n)\,\Delta^{\kappa_{r-1}}(n)\ \ldots\ \Delta^{\kappa_1}(n).$$

Now

$$\Delta^x u_n v_n = \Delta^x u_n \cdot v_n + x\Delta^{x-1} u_{n+1} \cdot \Delta v_n + \frac{x(x-1)}{\lfloor 2} \Delta^{x-2} u_{n+2} \cdot \Delta^2 v_n + \&c.$$

Hence, putting $x = \kappa_r$, $u_n = (n)$, and writing for shortness

$$\Delta^{\kappa_r}(n) = (n)_{\kappa_r}, \qquad \Delta^{\kappa_r - 1}(n+1) = (n+1)_{\kappa_r - 1},$$

$$\Delta^{\kappa_r - 2}(n+2) = (n+2)_{\kappa_r - 2}, \quad \&c.,$$

we have

$$\Delta^{\kappa_r}(n) v_n = (n)_{\kappa_r} v_n + \kappa_r (n+1)_{\kappa_r - 1} \Delta v_n + \frac{\kappa_r(\kappa_r - 1)}{\lfloor 2} (n+2)_{\kappa_r - 2} \Delta^2 v_n + \&c.;$$

and substituting for v_n, $\Delta^{\kappa_{r-1}}(n)\, \Delta^{\kappa_{r-2}}(n) \ldots \Delta^{\kappa_1}(n)$,
the general term of this expression is

$$\frac{\lfloor \kappa_r}{\lfloor l_{r-1} \lfloor \kappa_r - l_{r-1}} (n + l_{r-1})_{\kappa_r - l_{r-1}} \Delta^{\kappa_{r-1} + l_{r-1}}(n)\, \Delta^{\kappa_{r-2}}(n) \ldots \Delta^{\kappa_1}(n).$$

Continuing this process, the general term of $\phi(n, r)$ is found to be

$$\frac{(-)^r a^{rn + \kappa_1 + 2\kappa_2 + \ldots + r\kappa_r}}{(1-a)^{\kappa_1 + 1} (1-a^2)^{\kappa_2 + 1} \ldots (1-a^r)^{\kappa_r + 1}} \cdot \frac{\lfloor \kappa_r}{\lfloor l_{r-1} \lfloor \kappa_r - l_{r-1}}$$

$$\times \frac{\lfloor \kappa_{r-1} + l_{r-1}}{\lfloor l_{r-2} \lfloor \kappa_{r-1} + l_{r-1} - l_{r-2}} \cdot \frac{\lfloor \kappa_{r-2} + l_{r-2}}{\lfloor l_{r-3} \lfloor \kappa_{r-2} + l_{r-2} - l_{r-3}} \cdots\cdots \frac{\lfloor \kappa_2 + l_2}{\lfloor l_1 \lfloor \kappa_2 + l_2 - l_1}$$

$$\times (n + l_{r-1})_{\kappa_r - l_{r-1}} \cdot (n + l_{r-2})_{\kappa_{r-1} + l_{r-1} - l_{r-2}} \cdot (n + l_{r-3})_{\kappa_{r-2} + l_{r-2} - l_{r-3}} \cdots\cdots$$

$$\cdots\cdots (n + l_1)_{\kappa_2 + l_2 - l_1} \cdot (n)_{\kappa_1 + l_1}.$$

Now put

$$\left.\begin{aligned} \kappa_1 + l_1 &= \iota_1 \\ \kappa_2 + l_2 - l_1 &= \iota_2 \\ \kappa_3 + l_3 - l_2 &= \iota_3 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \\ \kappa_{r-1} + l_{r-1} - l_{r-2} &= \iota_{r-1} \\ \kappa_r - l_{r-1} &= \iota_r \end{aligned}\right\}.$$

Then

$$\begin{aligned} l_1 &= \iota_1 - \kappa_1, \\ l_2 &= \iota_1 + \iota_2 - \kappa_1 - \kappa_2, \\ &\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ l_{r-1} &= \iota_1 + \iota_2 + \ldots + \iota_{r-1} - \kappa_1 - \kappa_2 - \ldots - \kappa_{r-1}, \\ \iota_1 + \iota_2 + \ldots + \iota_r &= \kappa_1 + \kappa_2 + \ldots + \kappa_r; \end{aligned}$$

and the general term of $\phi(n, r)$ becomes

$$\frac{(-)^r a^{rn + \kappa_1 + 2\kappa_2 + \ldots + r\kappa_r}}{(1-a)^{\kappa_1 + 1} (1-a^2)^{\kappa_2 + 1} \ldots (1-a^r)^{\kappa_r + 1}} \frac{\lfloor l_1 + \iota_2 \lfloor l_2 + \iota_3 \ldots \lfloor l_{r-1} + \iota_r}{\lfloor l_1 \lfloor \iota_2 \lfloor l_2 \lfloor \iota_3 \ldots \lfloor l_{r-1} \lfloor \iota_r}$$

$$\times (n)_{\iota_1} (n + l_1)_{\iota_2} (n + l_2)_{\iota_3} \ldots (n + l_{r-1})_{\iota_r} \ldots\ldots (16).$$

The complete expression of which (16) is the general term is found by giving each of the quantities $\iota_1, \iota_2, \ldots \iota_r, \kappa_1, \kappa_2, \ldots \kappa_r$ all possible values [$l_1, l_2, \ldots l_{r-1}$ being given in terms of $\iota_1, \iota_2, \ldots \iota_r$ and $\kappa_1, \kappa_2, \ldots \kappa_r$], and adding the results.

Each of these quantities is either zero or a positive integer; and since (n) is a rational integral function of n of degree m, the greatest value of any one of the quantities $\iota_1, \iota_2, \ldots \iota_r$ is m.

Hence the greatest possible value of κ_1 is m, the greatest possible value of κ_2 is $2m$, and so on.

Thus each of the quantities $\iota_1, \iota_2, \ldots \iota_r$ has all positive integral values from 0 to m inclusive.

κ_1 has all positive integral values from 0 to m inclusive;
κ_2 „ „ „ 0 to $2m$ „
... ...
κ_r „ „ „ 0 to rm „

Those values of $\iota_1, \iota_2, \ldots \iota_r$ and $\kappa_1, \kappa_2, \ldots \kappa_r$ which make any of the quantities $l_1, l_2, \ldots l_{r-1}$ negative are inadmissible and must be rejected, as also those that do not satisfy the relation

$$\kappa_1+\kappa_2+\ldots+\kappa_r = \iota_1+\iota_2+\ldots+\iota_r.$$

For instance, when $r=2$, the general term of $\phi(n, 2)$ is

$$\frac{a^{2n+\kappa_1+2\kappa_2}}{(1-a)^{\kappa_1+1}(1-a^2)^{\kappa_2+1}} \frac{\lfloor l_1+\iota_2}{\lfloor l_1 \lfloor \iota_2} (n)_{\iota_1} (n+l_1)_{\iota_2};$$

and all the possible values of $\iota_1, \iota_2, \kappa_1, \kappa_2, l_1$ are found from the following table :—

$r=2$

ι_1	ι_2	κ_1	κ_2	l_1	
0	0	0	0	0	$m=1$
0	1	0	1	0	
1	0	0	1	1	
1	0	1	0	0	
1	1	0	2	1	
1	1	1	1	0	
0	2	0	2	0	$m=2$
1	2	0	3	1	
1	2	1	2	0	
2	0	0	2	2	
2	0	1	1	1	
2	0	2	0	0	
2	1	0	3	2	
2	1	1	2	1	
2	1	2	1	0	
2	2	0	4	2	
2	2	1	3	1	
2	2	2	2	0	
...	...	...	...	...	
...	...	...	...	...	
...	...		...	...	

Hence

$$\phi(n,2) = \frac{a^{2n}}{(1-a)(1-a^2)}\left\{(n)^2 + \frac{a^2}{1-a^2}(n)(n)_1 + \frac{a^2}{1-a^2}(n)_1(n+1) + \frac{a}{1-a}(n)_1(n) + \frac{a^4}{(1-a^2)^2}2(n)_1(n+1)_1 + \frac{a^3}{(1-a)(1-a^2)}(n)_1^2 + \ldots\right\}.$$

The terms given are those corresponding to $m = 1$.

Since
$$(n+l)_t = (1+\Delta)^l \Delta^t (n)$$
$$= (n)_t + l(n)_{t+1} + \frac{l(l-1)}{\lfloor 2}(n)_{t+2} + \ldots,$$

it follows that $\phi(n,r)$ is of the form

$$\frac{(-)^r a^{rn}}{(1-a)(1-a^2)\ldots(1-a^r)}[(n),(n)_1,(n)_2,\ldots(n)_m]^r.$$

For example, in the result given above, put $(n+1) = (n)+(n)_1$, $(n+1)_1 = (n)_1+(n)_2$, ...

Then

$$\phi(n,2) = \frac{a^{2n}}{(1-a)(1-a^2)}\left\{(n)^2 + (n)(n)_1\left[\frac{2a^2}{1-a^2} + \frac{a}{1-a}\right] + (n)_1^2\left[\frac{a^2}{1-a^2} + \frac{2a^4}{(1-a^2)^2} + \frac{a^3}{(1-a)(1-a^2)}\right] + \text{terms involving } (n)_2, (n)_3, \ldots\right\}$$

$$= \frac{a^{2n}}{(1-a)(1-a^2)}\left\{(n)^2 + (n)(n)_1\frac{a+3a^2}{1-a^2} + (n)_1^2\frac{a^2+a^3+2a^4}{(1-a^2)^2} + \ldots\right\}$$

$$= \frac{a^{2n}}{(1-a)(1-a^2)}[(n),(n)_1,(n)_2,\ldots(n)_m]^2.$$

The terms given are those corresponding to $m = 1$.

When $a=1$, $\phi(n,r) = \infty$, each element of the determinant (13) assumes the form $\pm\infty$, the value found for $C_r[(0),(1),\ldots(n-1)]$ is indeterminate, and the foregoing investigation fails.

And generally, when $a^x = 1$, where x is any integer from 1 to r inclusive, $\phi(n,x)$, $\phi(n,x+1)$, $\ldots$ $\phi(n,r)$ are all infinite, and $C_r[(0),(1)a,\ldots(n-1)a^{n-1}]$ is indeterminate.

In the simplest case, when $a = 1$, the general term of $C_r[(0),(1),\ldots(n-1)]$ may be found by the following method:

It will be shown hereafter that when (n) is expressed in the form

$$A_0 + A_1 n + A_2\frac{[n]^2}{\lfloor 2} + \ldots + A_m\frac{[n]^m}{\lfloor m},$$

and $\Sigma^p(n)$ is taken to be

$$A_0 \frac{[n]^p}{\lfloor p} + A_1 \frac{[n]^{p+1}}{\lfloor p+1} + \ldots + A_m \frac{[n]^{m+p}}{\lfloor m+p},$$

the constants of summation all vanish.

In the formula

$$\Sigma u_n v_n = u_{n-1} \cdot \Sigma v_n - \Delta u_{n-2} \cdot \Sigma^2 v_n + \Delta^2 u_{n-3} \cdot \Sigma^3 v_n - \&c.,$$

put $u_n = (n)$, and $v_n = C_{r-1}[(0), (1), \ldots (n-1)]$,

so that $\Sigma u_n v_n = C_r[(0), (1), \ldots (n-1)]$;

then, writing shortly

$$\Delta(n-2) = (n-2)_1, \quad \Delta^2(n-3) = (n-3)_2, \quad \ldots\ldots,$$

$$\begin{aligned} & C_r[(0), (1), \ldots (n-1)] \\ &= \{(n-1)\Sigma - (n-2)_1 \Sigma^2 + \ldots + (-)^m (n-m-1)_m \Sigma^{m+1}\} \\ &\qquad \times C_{r-1}[(0), (1), \ldots (n-1)] \\ &= \{(n-1)\Sigma - (n-2)_1 \Sigma^2 + \ldots\}^2 C_{r-2}[(0), (1), \ldots (n-1)] \\ &= \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ &= \{(n-1)\Sigma - (n-2)_1 \Sigma^2 + \ldots + (-)^m (n-m-1)_m \Sigma^{m+1}\}^{r-1} \Sigma(n), \end{aligned}$$

since $C_1[(0), (1), \ldots (n-1)] = \Sigma(n)$.

The general term of $C_r[(0), (1), \ldots (n-1)]$ is therefore

$$\begin{aligned} & (-)^{\kappa_1+1} (n-\kappa_1)_{\kappa_1-1} \Sigma^{\kappa_1} (-)^{\kappa_2+1} (n-\kappa_2)_{\kappa_2-1} \Sigma^{\kappa_2} \ldots\ldots \\ & \qquad \ldots\ldots (-)^{\kappa_{r-1}+1} (n-\kappa_{r-1})_{\kappa_{r-1}-1} \Sigma^{\kappa_{r-1}+1}(n), \end{aligned}$$

where each κ may have any integral value from 1 to $m+1$ inclusive.

It may easily be proved, by the same method as that given in Boole's "Finite Differences" (p. 74, Second Edition), that

$$\Sigma^\kappa u_n v_n = u_{n-\kappa} \cdot \Sigma^\kappa v_n - \kappa \Delta u_{n-\kappa-1} \cdot \Sigma^{\kappa+1} v_n + \frac{\kappa(\kappa+1)}{\lfloor 2} \Delta^2 u_{n-\kappa-2} \cdot \Sigma^{\kappa+2} v_n - \ldots,$$

the general term being $(-)^l \dfrac{\lfloor \kappa+l-1}{\lfloor \kappa-1 \; \lfloor l}\, \Delta^l u_{n-\kappa-l} \cdot \Sigma^{\kappa-l} v_n$.

Thus $C_r[(0), (1), \ldots (n-1)]$ consists of a series of terms of the form

$$\begin{aligned} & (-)^{\kappa_1+1+\kappa_2+l_1+1} (n-\kappa_1)_{\kappa_1-1} \frac{\lfloor \kappa_1+l_1-1}{\lfloor \kappa_1-1 \;\; \lfloor l_1} (n-\kappa_2-\kappa_1-l_1)_{\kappa_2+l_1-1} \\ & \qquad \times \Sigma^{\kappa_2+\kappa_1+l_1} (-)^{\kappa_3+1} (n-\kappa_3)_{\kappa_3-1} \ldots\ldots . \end{aligned}$$

Proceeding in this manner, the general term of $C_r[(0), (1), \ldots (n-1)]$ is found to be

$$(-)^{\kappa_1+\kappa_2+\dots+\kappa_{r-1}+l_1+l_2+\dots+l_{r-2}+r-1}$$
$$\times \frac{\lfloor \kappa_1+l_1-1 \quad \lfloor \kappa_2+\kappa_1+l_1+l_2-1 \quad \dots\dots \ (r-2) \text{ factors}}{\lfloor \kappa_1-1 \quad \lfloor l_1 \quad \lfloor \kappa_2+\kappa_1+l_1-1 \quad \lfloor l_2 \quad \dots\dots \quad \dots \quad \dots \quad \dots}$$
$$\times (n-\kappa_1)_{\kappa_1-1} \cdot (n-\kappa_2-\kappa_1-l_1)_{\kappa_2+l_1-1} \cdot (n-\kappa_3-\kappa_2-\kappa_1-l_1-l_2)_{\kappa_3+l_2-1} \dots\dots$$
$$\dots\dots \ (r-1) \text{ factors } \Sigma^{1+\kappa_1+\kappa_2+\dots\kappa_{r-1}+l_1+l_2+\dots l_{r-2}}(n).$$

Now put

$$\left.\begin{aligned} \kappa_1 &= \iota_1 \\ \kappa_2 + l_1 &= \iota_2 \\ \kappa_3 + l_2 &= \iota_3 \\ \dots \quad &\dots \quad \dots \\ \kappa_{r-1} + l_{r-2} &= \iota_{r-1} \end{aligned}\right\};$$

and let

$$\begin{aligned} \mathrm{I}_1 &= \iota_1, \\ \mathrm{I}_2 &= \iota_1 + \iota_2, \\ \mathrm{I}_3 &= \iota_1 + \iota_2 + \iota_3, \\ \dots \quad &\dots \quad \dots \quad \dots \\ \mathrm{I}_{r-1} &= \iota_1 + \iota_2 + \dots + \iota_{r-1}. \end{aligned}$$

Then the general term of $\mathrm{C}_r[(0), (1), \dots (n-1)]$ becomes

$$(-)^{\mathrm{I}_{r-1}+r-1} \frac{\lfloor \mathrm{I}_1+l_1-1 \lfloor \mathrm{I}_2+l_2-1 \dots \lfloor \mathrm{I}_{r-2}+l_{r-2}-1}{\lfloor \mathrm{I}_1-1 \lfloor l_1 \lfloor \mathrm{I}_2-1 \lfloor l_2 \dots \lfloor \mathrm{I}_{r-2}-1 \lfloor l_{r-2}}$$
$$\times (n-\mathrm{I}_1)_{\iota_1-1} \cdot (n-\mathrm{I}_2)_{\iota_2-1} \dots (n-\mathrm{I}_{r-1})_{\iota_{r-1}-1} \cdot \Sigma^{\mathrm{I}_{r-1}+1}(n).$$

Different terms for which $\iota_1, \iota_2, \dots \iota_{r-1}$ have respectively the same values differ only in their coefficients, and may be added together. The general coefficient of the sum of such terms is found by giving $l_1, l_2, \dots l_{r-2}$ all possible values in the coefficient of the general term, and adding the results.

Now, since the least value of any of the quantities $\kappa_1, \kappa_2, \dots \kappa_{r-1}$ is 1, the greatest value of l_1 is ι_2-1, the greatest value of l_2 is ι_3-1, and so on. Thus the coefficient becomes

$$\left\{1+\mathrm{I}_1+\frac{\mathrm{I}_1(\mathrm{I}_1+1)}{\lfloor 2}+\dots \iota_2 \text{ terms}\right\}\left\{1+\mathrm{I}_2+\frac{\mathrm{I}_2(\mathrm{I}_2+1)}{\lfloor 2}+\dots \iota_3 \text{ terms}\right\}\dots$$

which may easily be shewn equal to

$$\frac{\lfloor \mathrm{I}_1+\iota_2-1}{\lfloor \mathrm{I}_1 \ \lfloor \iota_2-1} \cdot \frac{\lfloor \mathrm{I}_2+\iota_3-1}{\lfloor \mathrm{I}_2 \ \lfloor \iota_3-1} \dots = \frac{\lfloor \mathrm{I}_2-1}{\lfloor \mathrm{I}_1 \ \lfloor \iota_2-1} \cdot \frac{\lfloor \mathrm{I}_3-1}{\lfloor \mathrm{I}_2 \ \lfloor \iota_3-1} \dots \frac{\lfloor \mathrm{I}_{r-1}-1}{\lfloor \mathrm{I}_{r-2} \ \lfloor \iota_{r-1}-1}.$$

And since there are $r-1$ of the quantities $\iota_1, \iota_2, \dots \iota_{r-1}$, each of which may have any positive integral value from 1 to $n+1$ inclusive.

$C_r[(0), (1), \ldots (n-1)]$ consists of $(m+1)^{r-1}$ terms of the form

$$(-)^{I_{r-1}+r-1}\frac{\lfloor I_2-1}{\lfloor I_1 \lfloor \iota_2-1}\cdot\frac{\lfloor I_3-1}{\lfloor I_2 \lfloor \iota_3-1}\cdots\cdots\frac{\lfloor I_{r-1}-1}{\lfloor I_{r-2} \lfloor \iota_{r-1}-1}$$

$$\times(n-I_1)_{\iota_1-1}\cdot(n-I_2)_{\iota_2-1}\ldots(n-I_{r-1})_{\iota_{r-1}-1}\cdot\Sigma^{I_{r-1}+1}(n)\ldots.(17).$$

The degree of each of these terms is

$$(m-\iota_1+1)+(m-\iota_2+1)+\ldots+(m-\iota_{r-1}+1)+(m+I_{r-1}+1)=(m+1)r.$$

The constants of summation, being found from the condition that $C_r(a_0, a_1, \ldots a_{n-1})=0$, when $n=0, 1, \ldots r-1$; all vanish. For each term contains a factor of the form $\Sigma^{I_{r-1}+1}(n)$, the least value of I_{r-1} being $r-1$; and by hypothesis

$$\Sigma^p(n)=A_0\frac{[n]^p}{\lfloor p}+A_1\frac{[n]^{p+1}}{\lfloor p+1}+\ldots\ldots,$$

which, when $p=r$, and for all higher values of p, vanishes when n is put $=0, 1, \ldots r-1$.

For the same reason the constants of summation all vanish when

$$(n)=A_0+A_1[n+1]+A_2\frac{[n+1]^2}{\lfloor 2}+\ldots\ldots,$$

$$(n)=A_0+A_1[n+1]+A_2\frac{[n+2]^2}{\lfloor 2}+\ldots\ldots.$$

Provided that $\Sigma^p(n)$ is taken to be

$$A_0[n]^p+A_1\frac{[n+1]^{p+1}}{\lfloor p+1}+A_2\frac{[n+1]^{p+2}}{\lfloor p+2}+\ldots\ldots,$$

$$A_0[n]^p+A_1\frac{[n+1]^{p+1}}{\lfloor p+1}+A_2\frac{[n+2]^{p+2}}{\lfloor p+2}+\ldots\ldots.$$

Let $r=2$, then the general term of $C_2[(0), (1), \ldots (n-1)]$ is

$$(-)^{\iota_1+1}(n-\iota_1)_{\iota_1-1}\cdot\Sigma^{\iota_1+1}(n).$$

Thus $C_2[(0), (1), \ldots (n-1)]$

$$=(n-1)\Sigma^2(n)-(n-2)_1\Sigma^3(n)+\ldots+(-)^m(n-m-1)_m\Sigma^{m+1}(n)\ldots(18);$$

this result is a particular case of the well known formula for the sum of the product of two functions; viz., the functions are (n) and $\Sigma(n)$.

In the case $(n)=n+1$,

$$C_2(1, 2, \ldots n)=n\cdot\frac{[n+1]^3}{\lfloor 3}-1\cdot\frac{[n+1]^4}{\lfloor 4}$$

$$=\frac{n+1\cdot n\cdot n-1\cdot 3n+2}{24}\ldots\ldots\ldots\ldots(19).$$

In the case $(n) = [n+1]^2 = n+1+\frac{2[n+1]^2}{\lfloor 2}$,

$$C_2(1, 4, \ldots n^2)$$

$$= n^2\left\{\frac{[n+1]^3}{\lfloor 3} + \frac{2[n+1]^4}{\lfloor 4}\right\} - \frac{}{2n-1}\left\{\frac{[n+1]^4}{\lfloor 4} + \frac{2[n+1]^5}{\lfloor 5}\right\}$$

$$+ 2\left\{\frac{[n+1]^5}{\lfloor 5} + \frac{2[n+1]^6}{\lfloor 6}\right\};$$

which reduces to

$$C_2(1, 4, \ldots n^2) = \frac{n+1 \,.\, n \,.\, n-1 \,.\, 2n+1 \,.\, 2n-1 \,.\, 5n+6}{360} \quad \ldots(20).$$

When $r=3$ the general term becomes

$$(-)^{I_2}\frac{\lfloor I_2-1}{\lfloor I_1 \lfloor \iota_2-1}\,(n-I_1)_{\iota_1-1}\,.\,(n-I_2)_{\iota_2-1}\,.\,\Sigma^{I_2+1}(n).$$

Thus, writing down the corresponding terms of $C_3[(0), (1), \ldots (n-1)]$, and values of ι_1, ι_2, I_1, I_2,

ι_1	ι_2	I_1	I_2		
1	1	1	2	$(n-1)\ (n-2)\ \Sigma^3(n)$	$= C_3[(0), (1), \ldots (n-1)]$ when $m=1$ ..... (21);
1	2	1	3	$-\ 2\,(n-1)\ (n-3)_1\,\Sigma^4(n)$	
2	1	2	3	$-\ (n-2)_1\,(n-3)\ \Sigma^4(n)$	
2	2	2	4	$+\ 3\,(n-2)_1\,(n-4)_1\,\Sigma^5(n)$	
1	3	1	4	$+\ 3\,(n-1)\ (n-4)_2\,\Sigma^5(n)$	$= C_3[(0), (1), \ldots (n-1)]$ when $m=2$ ..... (22).
3	1	3	4	$+\ (n-3)_2\,(n-4)\ \Sigma^5(n)$	
2	3	2	5	$-\ 6\,(n-2)_1\,(n-5)_2\,\Sigma^6(n)$	
3	2	3	5	$-\ 4\,(n-3)_2\,(n-5)_1\,\Sigma^6(n)$	
3	3	3	6	$+10\,(n-3)_2\,(n-6)_2\,\Sigma^7(n)$	
...	...	...	...	... ...	
...	...	...	...	... ...	

When $r=4$ the general term becomes

$$(-)^{I_3+1}\frac{\lfloor I_2-1}{\lfloor I_1 \lfloor \iota_2-1}\,.\,\frac{\lfloor I_3-1}{\lfloor I_2 \lfloor \iota_3-1}$$

$$\times\,(n-I_1)_{\iota_1-1}\,.\,(n-I_2)_{\iota_2-1}\,.\,(n-I_3)_{\iota_3-1}\,.\,\Sigma^{I_3+1}(n).$$

Thus, writing down the corresponding terms of $C_4[(0), (1), \ldots (n-1)]$, and values of ι_1, ι_2, ι_3, I_1, I_2, I_3,

ι_1	ι_2	ι_3	I_1	I_2	I_3	
1	1	1	1	2	3	$(n-1)\ (n-2)\ (n-3)\ \Sigma^4(n)$
1	1	2	1	2	4	$-\ 3\,(n-1)\ (n-2)\ (n-4)_1\,\Sigma^5(n)$
1	2	1	1	3	4	$-\ 2\,(n-1)\ (n-3)_1\,(n-4)\ \Sigma^5(n)$
2	1	1	2	3	4	$-\ (n-2)_1\,(n-3)\ (n-4)\ \Sigma^5(n)$
1	2	2	1	3	5	$+\ 8\,(n-1)\ (n-3)_1\,(n-5)_1\,\Sigma^6(n)$
2	1	2	2	3	5	$+\ 4\,(n-2)_1\,(n-3)\ (n-5)_1\,\Sigma^6(n)$
2	2	1	2	4	5	$+\ 3\,(n-2)_1\,(n-4)_1\,(n-5)\ \Sigma^6(n)$
2	2	2	2	4	6	$-15\,(n-2)_1\,(n-4)_1\,(n-6)_1\,\Sigma^7(n)$
...	...	..	...	...	...	... ...
...	.	..	...	...	...	... ...

$= C_4[(0), (1), \ldots (n-1)]$ when $m=1$ ..... (23).

The following results are obtained from formulæ (18), (21), (22), and (23). I have not given proofs of them, as they are obtained, like (19) and (20), by common algebraical substitution and reduction; but I have verified them carefully, and believe them to be free from error:

$$C_2(1, 2, \ldots n) = \frac{n+1\,.\,n\,.\,n-1\,.\,3n+2}{24},$$

$$C_3(1, 2, \ldots n) = \frac{(n+1)^2\,n^2\,(n-1)\,(n-2)}{48},$$

$$C_4(1, 2, \ldots n) = \frac{n+1\,.\,n\,.\,n-1\,.\,n-2\,.\,n-3}{8\,\lfloor 6} (15n^3+15n^2-10n-8),$$

$$C_2(1, 4, \ldots n^2) = \frac{n+1\,.\,n\,.\,n-1\,.\,2n+1\,.\,2n-1\,.\,5n+6}{3\,\lfloor 5},$$

$$C_3(1, 4, \ldots n^2) = \frac{n+1\,.\,n\,.\,n-1\,.\,n-2\,.\,2n+1\,.\,2n-1\,.\,2n-3}{9\,\lfloor 7} \times (35n^2+91n+60).$$

In the last two formulæ it is remarkable that

$C_2(1, 4, \ldots n^2)$ is divisible by $2n+1\,.\,2n-1$; *i.e.*, by $\Delta n^2\,.\,\Delta(n-1)^2$.

$C_3(1, 4, \ldots n^2)$ is divisible by $\Delta n^2\,.\,\Delta(n-1)^2\,.\,\Delta(n-2)^2$.

It is well known that $C_1(1, 4, \ldots n^2) = \frac{n+1\,.\,n\,.\,2n+1}{6}$, which is divisible by Δn^2; but I do not know whether similar results hold for $C_4(1, 4, \ldots n^2)$ ... or not.

Since $C_r(a_0, a_1, \ldots a_{n-1}) = \left(\sum_0^n a_n\right)^r$, $C_r[(0), (1), \ldots (n-1)]$ may be found by a repeated use of the Euler-Maclaurin sum formula.

For example,

$$\begin{aligned} C_2(1, 2, \ldots n) &= \left(\sum_0^n \overline{n+1}\right)^2 \\ &= \sum_0^n \overline{n+1}\left(\frac{n^2}{2} + n - \frac{n}{2}\right) \\ &= \sum_0^n \left(\frac{n^3}{2} + n^2 + \frac{n}{2}\right) \\ &= \left(\frac{n^4}{8} + \frac{n^3}{3} + \frac{n^2}{4}\right) - \frac{1}{2}\left(\frac{n^3}{2} + n^2 + \frac{n}{2}\right) \\ &\qquad + \frac{1}{12}\left(\frac{3n^2}{2} + 2n\right) \\ &= \frac{3n^4+2n^3-3n^2-2n}{24} = \frac{n+1 \,.\, n \,.\, n-1 \,.\, 3n+2}{24}. \end{aligned}$$

Since reading this paper I have re-written it in accordance with some valuable suggestions, for which I am indebted to Prof. Cayley, and have added to it considerably.

March 10th, 1876.

W. SPOTTISWOODE, Esq., F.R.S., Vice-President, in the Chair.

Messrs Arthur Cockshott and R. T. Wright were elected, and Messrs. E. B. Elliott, C. M. Leudesdorf, and J. W. Russell, were admitted into the Society.

The following communications were made to the Society :—

Prof. Cayley, "On the Bicursal Sextic, and the Problem of Three-Bar Motion."

Prof. Clifford, "On the Classification of Geometric Algebras."

Presents received :—

Extraits de deux lettres adressées à D. B. Boncampagni par M. le Comte Léopold Hugo (extrait des Atti dell' Accademia Pontificia de' Nuovi Lincei, 19 Dec., 1875).

"Sur une classe de points singuliers de surfaces," par H. G. Zeuthen (from the Math. Annalen, ix.), Copenhague, 27 Août, 1875.

Extrait d'une lettre de M. Ch. Hermite à M. L. Königsberger "Sur le développement des fonctions elliptiques suivant les puissances croissantes de la variable" (Crelle, 81 vol.)

Lettre de M. Ch. Hermite à M. Borchardt "Sur la fonction de Jacob Bernoulli" (Crelle, vol. 79).

Extrait d'une lettre de M. Ch. Hermite à M. Borchardt "Sur la

transformation des formes quadratiques ternaires en elles-mêmes" (Crelle, 78 vol.)

Extrait d'une lettre de M. Ch. Hermite à M. Borchardt "Sur la réduction des formes quadratiques ternaires" (Crelle, vol. 79).

The same to the same on a property of Bernoulli's numbers (Crelle, vol. 81).

"Sur les développements de la fonction F $(x) = sn^a x \,.\, cn^b x \,.\, dn^c x$, où les exposants sont entiers," par M. Ch Hermite (Bihang till K. Svenska vet Akad. Handlingar. Band 3, No. 10), Stockholm, 1875.

Extrait d'une lettre de M. Ch. Hermite de Paris à M. L. Fuchs de Gottingue "Sur quelques équations différentielles linéaires" (Crelle, vol. 79).

(*The above presented by M. Ch. Hermite.*)

"Nouvelles propriétés géométriques de la surface de l'onde qui s'interprètent en Optique," par M. A. Mannheim, 7 Février, 1876.

"Démonstration géometrique d'une relation due à M. Laguerre" (the same, 6 Mars, 1876).

"Association Française pour l'avancement des Sciences—Congrès de Nantes, Paris, 1875;" containing "Recherches sur la surface de l'onde (21 Août 1875), "Propriétés des diamètres de la surface de l'onde et interprétation physique de ces propriétés" (25 Août, 1875).

(*The above presented by M. Mannheim.*)

Carte de Visite likeness and photograph, from Dr. H. G. Zeuthen.

On Three-Bar Motion. By Prof. Cayley.

[*Read March 10th*, 1876.]

The discovery by Mr. Roberts of the triple generation of a Three-Bar Curve, throws a new light on the whole theory, and is a copious source of further developments.* The present paper gives in its most simple form the theorem of the triple generation; it also establishes the relation between the nodes and foci; and it contains other researches. I have made on the subject a further investigation, which I give in a separate paper, "On the Bicursal Sextic"; but the two papers are intimately related, and should be read in connection.

The Three-Bar Curve is derived from the motion of a system of three bars of given lengths pivoted to each other, and to two fixed

* See his paper "On Three-Bar Motion in Plane Space," *ante*, pp. 15–23, which contains more than I had supposed of the results here arrived at. There is no question as to Mr. Roberts' priority in all his results.

points, so as to form the three sides of a quadrilateral, the fourth side of which is the line joining the two fixed points; the curve is described by a point rigidly connected with the middle bar; or, what is more convenient, we take the middle bar to be a triangle pivoted at the extremities of the base to the other two bars (say, the radial bars), and having its vertex for the describing point.

Including the constants of position and magnitude, the Three-Bar Curve thus depends on nine parameters; viz., these are the coordinates of the two fixed points, the lengths of the connecting bars, and the three sides of the triangle. It is known that the curve is a tricircular trinodal sextic, and the equation of such a curve contains $27-6-6-3, = 12$ constants. Imposing on the curve the condition that the three nodes lie upon a given curve, the number of constants is reduced to $12-3, = 9$: and it is in this way that the Three-Bar Curve is distinguished from the general tricircular trinodal sextic; viz., in the Three-Bar Curve the two fixed points are foci, and they determine a third focus;* and the condition is that the nodes are situate on the circle through the 3 foci.

The nodes are two of them arbitrary points on the circle, and the third of them is a point such that (measuring the distances along the circle from any fixed point of the circumference) the sum of the distances of the nodes is equal to the sum of the distances of the foci. Considering the two fixed points as given, the curve depends upon five parameters, viz., the lengths of the connecting bars and the sides of the triangle. Taking the *form* of the triangle as given, there are then only three parameters, say the lengths of the connecting bars and the base of the triangle; in this case the third focus is determined, and therefore the circle through the three foci; we may then take two of the nodes as given points on this circle, and thereby establish two relations between the three parameters (in fact, we thereby determine the differences of the squares of the lengths in question): but the third node is then an absolutely determined point on the circle, and we cannot make use of it for completing the determination of the parameters; viz., one parameter remains arbitrary. Or, what is the same thing, given the three foci and also the three nodes (consistently with the foregoing conditions, viz., the nodes lie in the centre through the three foci, the sum of the distances of the nodes being equal to the sum of the distances of the foci) we have a singly infinite series of three-bar curves.

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* A focus is a point, the intersection of a tangent to the curve from one circular point at infinity with a tangent from the other circular point at infinity; if the circular points are simple or multiple points on the curve, then the tangent or tangents *at* a circular point should be excluded from the tangents *from* the point; and the intersection of two such tangents at the two circular points respectively is not an ordinary focus; but as the points in question are the only kind of foci occurring in the present paper, I have in the text called them foci.

In reference to the notation proper for the theorem of the triple generation, I shall, when only a single node of generation is attended to, take the curve to be generated as shown in the annexed Figure 1; viz., O is the generating point, OC_1B_1 the triangle, C, B the fixed points, CC_1 and BB_1 the radial bars. The sides of the triangle are a_1, b_1, c_1; its angles are O, = A, B_1, = B, C_1, = C: the bars CC_1 and BB_1 are $= a_2$ and a_3 respectively, and the distance CB is $= a$. The sides a_1, b_1, c_1 may be put $= k_1$ (sin A, sin B, sin C), and the lines a_1, a_2, a_3 = sin A (k_1, k_2, k_3), viz., the original data a_1, b_1, c_1, a_1, a_2, a_3, may be replaced by the angles A, B, C ($A+B+C=\pi$) and the lines k, k_1, k_2, k_3. And it is convenient to mention at once that the third focus A is then a point such that ABC is a triangle similar and congruent to OB_1C_1.

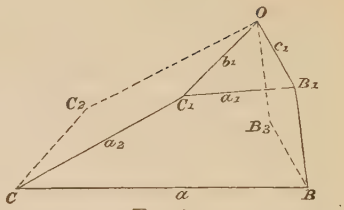

Fig. 1.

It may be remarked that, producing CC_1 and BB_1 to meet in a point α, this is the centre of instantaneous rotation of the triangle, and therefore αO the normal to the curve at O.

I proceed to show that the three nodes F, G, H are in the circle circumscribed about ABC, and that their positions are such that (the distances being measured along the circle as before) we have, Sum of the distances of F, G, H equal Sum of the distances of A, B, C.

Supposing O to be at a node F, we have then the two equal triangles FB_1C_1, $FB_1'C_1'$ such that C_1, C_1' are equidistant from C, and B_1, B_1' equidistant from B. Hence the angles B_1FB_1', C_1FC_1' are equal; consequently the halves of these angles CFC_1' and BFB_1' are equal; whence angle CFB is equal to angle $C_1'FB_1'$, that is, to the angle A; or F lies on a circle through B, C such that the segment upon BC contains the angle A, that is, upon the circle through A, B, C. To complete the investigation of the nodes, suppose $CF = \tau$, $BF = \sigma$: then the condition $\angle CFC_1' = \angle BFB_1'$ gives

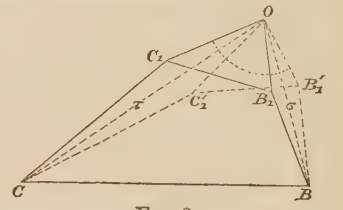

Fig. 2.

$$\frac{b_1^2+\tau^2-a_2^2}{2b_1\tau} = \frac{c_1^2+\sigma^2-a_3^2}{2c_1\sigma},$$

that is, $$c_1\sigma\,(b_1^2+\tau^2-a_2^2) - b_1\tau\,(c_1^2+\sigma^2-a_3^2) = 0;$$

and the condition that F is on the circle gives

$$\sigma^2+\tau^2-2\sigma\tau\cos A = a^2.$$

These equations give six values of (σ, τ) corresponding in pairs to each other; viz., if (σ_1, τ_1) is a solution, then $(-\sigma_1, -\tau_1)$ is also a solution; and to each pair of solutions corresponds a single point on the circle, viz., we have thus the three nodes F, G, H.

Writing the foregoing equation in the form

$$\{c_1(b_1^2-a_2^2)\sigma - b_1(c_1^2-a_3^2)\tau\}(\sigma^2+\tau^2-2\sigma\tau\cos A) + a^2(c_1\sigma\tau^2 - b_1\sigma^2\tau) = 0,$$

and putting the left-hand side $= M(\sigma-p_1\tau)(\sigma-p_2\tau)(\sigma-p_3\tau)$; then, if α, β, γ denote $\cos A+i\sin A$, $\cos B+i\sin B$, $\cos C+i\sin C$ respectively, putting first $\sigma=\alpha\tau$ and next $\sigma=\frac{\tau}{\alpha}$, and dividing one of these results by the other, we find

$$\frac{c_1-b_1\alpha}{c_1\alpha-b_1} = \frac{\alpha-p_1\,.\,\alpha-p_2\,.\,\alpha-p_3}{1-p_1\alpha\,.\,1-p_2\alpha\,.\,1-p_3\alpha}.$$

The left-hand side is here

$$= \frac{\sin C-\alpha\sin B}{\alpha\sin C-\sin B}, = \frac{\sin C-\sin B(\cos A+i\sin A)}{\sin C(\cos A+i\sin A)-\sin B},$$

$$= \frac{\sin A(\cos B-i\sin B)}{-\sin A(\cos C-i\sin C)}, = -\frac{\gamma}{\beta},$$

or the equation is

$$\frac{\alpha-p_1\,.\,\alpha-p_2\,.\,\alpha-p_3}{1-p_1\alpha\,.\,1-p_2\alpha\,.\,1-p_3\alpha} = -\frac{\gamma}{\beta}.$$

Also, writing f for the angle FCB, we have $\sigma = \frac{\sin f}{\sin(A+f)}\tau$, viz., the values of p_1, p_2, p_3 are $\frac{\sin f}{\sin(A+f)}$, $\frac{\sin g}{\sin(A+g)}$, $\frac{\sin h}{\sin(A+h)}$. We thence find

$$\frac{\alpha-p_1}{1-\alpha p_1} = \frac{\sin(A+f)(\cos A+i\sin A)-\sin f}{\sin(A+f)-(\cos A+i\sin A)\sin f}, = \frac{\cos(A+f)+i\sin(A+f)}{\cos f-i\sin f},$$

$$= \cos(A+2f)+i\sin(A+2f);$$

with the like values for the other two values. Hence, writing also $-\frac{\gamma}{\beta} = -\cos(C-B)-i\sin(C-B), = \cos(\pi+C-B)+i\sin(\pi+C-B)$, the equation becomes

$$\cos(3A+2f+2g+2h)+i\sin(3A+2f+2g+2h) = \cos(\pi+C-B)+i\sin(\pi+C-B),$$

that is, $3A+2f+2g+2h = \pi+C-B$,

or, what is the same thing,

$$2f+2g+2h = \pi+C-B-3A.$$

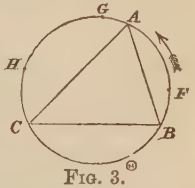

Fig. 3.

Reckoning the angles round the centre from a point Θ on the circumference, if A′, B′, C′, F′, G′, H′ are the angles belonging to the points A, B, C, F, G, H respectively, then

$$A' = \lambda+2C, \qquad F' = \lambda+2f,$$
$$B' = \lambda, \qquad G' = \lambda+2g,$$
$$C' = \lambda+2C+2B, \quad H' = \lambda+2h;$$

and therefore

$$A'+B'+C' = 3\lambda+4C+2B, \quad F'+G'+H' = 3\lambda+2f+2g+2h,$$
$$= 3\lambda+\pi+C-B-3A;$$

that is, $A'+B'+C'-F'-G'-H' = -\pi+3(A+B+C)$; or, omitting an angle 2π, this is $A'+B'+C' = F'+G'+H'$, the equation which determines the relation between the three nodes on the circle ABC.

Reverting to the equation $c_1\sigma\,(b_1^2+\tau^2-a_2^2)-b_1\tau\,(c_1^2+\sigma^2-a_3^2)=0$, which belongs to a node, if we consider the form of the triangle as given, and write b_1, $c_1 = k_1 \sin B$, $k_1 \sin C$, this becomes

$$\sigma \sin C\,(b_1^2-a_2^2)-\tau \sin B\,(c_1^2-a_3^2)+\sigma\tau(\tau \sin C-\sigma \sin B)=0;$$

viz., considering the node as given, then the values of σ, τ are given, and the equation establishes a relation between the values of $b_1^2-a_2^2$ and $c_1^2-a_3^2$.* If a second node be given, we have a second relation between these same quantities, and the two equations give the values of the two quantities, viz., the values of $k_1^2 \sin^2 B - k_2^2 \sin^2 A$, $k_1^2 \sin^2 C - k_3^2 \sin^2 A$, or, what is the same thing, the value of $\dfrac{k_1^2}{\sin^2 A}-\dfrac{k_2^2}{\sin^2 B}$, $\dfrac{k_1^2}{\sin^2 A}-\dfrac{k_3^2}{\sin^2 B}$. It thus appears that, if l_1, l_2, l_3 are any values of k_1, k_2, k_3 belonging to a given system of three nodes, the general values of k_1, k_2, k_3 belonging to the same system of three nodes are

$$k_1^2 = l_1^2 + u \sin^2 A, \quad k_2^2 = l_2^2 + u \sin^2 B, \quad k_3^2 = l_3^2 + u \sin^2 C,$$

where u is an arbitrary constant.

It may be added that there will be a node at B, if the equation is satisfied by $\tau = 0$, $\sigma = a$, the condition is $b_1^2 - a_2^2 = 0$, that is, if $\dfrac{k_1}{\sin A} = \dfrac{k_2}{\sin B}$: similarly there will be a node at C, if $c_1^2 - a_3^2 = 0$, that is, if $\dfrac{k_1}{\sin A} = \dfrac{k_3}{\sin B}$; and a node at A, if $\dfrac{k_2}{\sin C} = \dfrac{k_3}{\sin B}$. If two of these equations are satisfied, the third equation is also satisfied, viz., we then have $\dfrac{k_1}{\sin A} = \dfrac{k_2}{\sin B} = \dfrac{k_3}{\sin C}$; and the three nodes coincide with the three foci respectively.

If, in Figure 2, the points C_1, C_1' coincide on the line CF, and therefore also the points B_1, B_1' coincide on the line BF, then, instead of a node at F, we have a cusp. We have in this case a triangle the sides of which are a_2+b_1, a_3+c_1, a, and the included angle between the first

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* Considering, in the equation, a_2 and a_3 as the distances of a variable point P from the points C and B respectively, the equation represents a circle having its centre on the line CB. Similarly, when a second node is given, the corresponding equation represents another circle, having its centre on the line CB, and the intersections of the two circles determine a_2 and a_3, the lengths of the radial bars, in order that the curve may have the given nodes.

two sides is $= A$: we have, therefore, the relation

$$a^2 = (a_2+b_1)^2+(a_3+c_1)^2-2\,(a_2+b_1)\,(a_3+c_1)\cos A.$$

Substituting herein for a, b, &c., the values $k \sin A$, $k \sin B$, &c., the equation is

$$k^2 \sin^2 A = (k_1 \sin B+k_2 \sin A)^2+(k_1 \sin C+k_3 \sin A)^2 \\ -2\,(k_1 \sin B+k_2 \sin A)\,(k_1 \sin C+k_3 \sin A)\cos A,$$

and expanding the right-hand side and reducing by means of $A+B+C = \pi$, the whole becomes divisible by $\sin^2 A$, and we have

$$k^2 = k_1^2+k_2^2+k_3^2-2k_2k_3\cos A+2k_3k_1\cos B+k_1k_2\cos C;$$

viz., considering A, B, C, k_1, k_2, k_3 as given, this equation determines k so that the curve may have a cusp. The equation is one of a system of four equations,

$$k^2 = k_1^2+k_2^2+k_3^2-2k_2k_3\cos A+2k_3k_1\cos B+2k_1k_2\cos C,$$
$$k^2 = k_1^2+k_2^2+k_3^2+2k_2k_3\cos A-2k_3k_1\cos B+2k_1k_2\cos C,$$
$$k^2 = k_1^2+k_2^2+k_3^2+2k_2k_3\cos A+2k_3k_1\cos B-2k_1k_2\cos C,$$
$$k^2 = k_1^2+k_2^2+k_3^2-2k_2k_3\cos A-2k_3k_1\cos B-2k_1k_2\cos C,$$

which belong to the different arrangements CC_1F or CFC_1, BB_1F or BFB_1, of the three points on the lines BF and CF; if k has any of these four values, the curve will have a cusp. If two of the equations subsist together, we have a curve with two cusps. Taking k_1, k_2, k_3, and also cos A, cos B, cos C, as positive (viz., assuming that the triangle is acute-angled), the fourth equation cannot subsist with any one of the others: but two of the others may subsist together, for instance, the first and second will do so, if $k_2k_3\cos A = k_3k_1\cos B$ $\left(\text{that is, } \frac{k_1}{\cos A} = \frac{k_2}{\cos B}\right)$, $k^2 = k_1^2+k_2^2+k_3^2+2k_1k_2\cos C$: the curve has then two cusps. Similarly, the three equations may subsist together, viz., we must then have

$$\frac{k_1}{\cos A} = \frac{k_2}{\cos B} = \frac{k_3}{\cos C},\quad k^2 = k_1^2+k_2^2+k_3^2+2k_2k_3\cos A,$$

and, writing herein k_1, k_2, $k_3 = \lambda \cos A$, $\lambda \cos B$, $\lambda \cos C$, we find

$$k^2 = \lambda^2\,(\cos^2 A+\cos^2 B+\cos^2 C+2\cos A\cos B\cos C) = \lambda^2;$$

viz., if k_1, k_2, k_3 are respectively $= k \cos A$, $k \cos B$, $k \cos C$, the curve has then three cusps. It will be recollected that, if

$$k_1 : k_2 : k_3 = \sin A : \sin B : \sin C,$$

the nodes coincide with the foci; the two sets of conditions subsist together if $A = B = C = 60°$; $k_1 = k_2 = k_3 = \frac{1}{2}k$, viz., we have then a curve with three cusps coinciding with the three foci respectively.

Before going further, I will establish the theorem for the triple generation of the curve.

The theorem which gives the triple generation may be stated as follows. See Figures 4, 5, 6.*

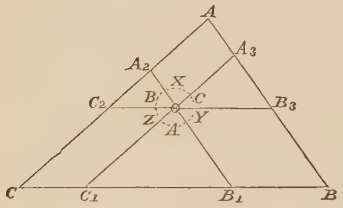


Fig. 4.

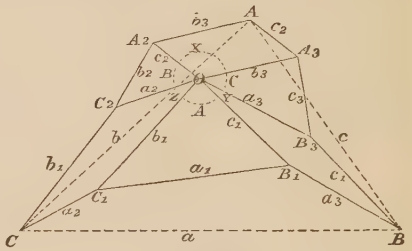


Fig. 5.

Imagine a triangle ABC and a point O, through which point are drawn lines parallel to the sides dividing the triangle into three triangles OB_1C_1, OC_2A_2, OA_3B_3, similar *inter se* and to the original triangle, and into three parallelograms OA_2AA_3, OB_3BB_1, OC_1CC_2: then, considering the three triangles as pivoted together at the point O, and replacing the exterior sides of the parallelograms by pairs of bars A_2AA_3, B_3BB_1, C_1CC_2 pivoted together at A, B, C, and to the triangles at A_2, A_3, B_3, B_1, C_1, C_2, the figure thus consisting of the three triangles and the six bars; let the three triangles be turned at pleasure about the point O, so as to displace in any manner the points A, B, C: we have the theorem that the triangle ABC will remain always similar to the original triangle ABC (that is, to each of the three triangles OB_1C_1, OC_2A_2, OA_3B_3): and further, that, starting from any given positions of the three triangles, we may so move them as not to alter the triangle ABC in magnitude: whence, conversely, fixing the three points A, B, C, the point O will be moveable in a curve.

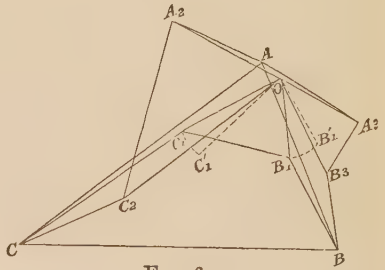


Fig. 6.

Assuming this, it is clear that the locus of the point O is simultaneously the locus given by

* Figure 6 (substantially the same as Fig. 5) belongs to the same curve as Figures 1 and 2, and it exhibits the triple generation of this curve: the generating point O being taken at a node (the same node as in Figure 2), and the two positions OB_1C_1 and $OB_1'C_1'$ of one of the triangles being shown in the figure.

Triangle OB_1C_1 connected by bars B_1B and C_1C to fixed points B, C,
„ OC_2A_2 „ C_2C „ A_2A „ C, A,
„ OA_3B_3 „ A_3A „ B_3B „ A, B;

or, that we have a triple generation of the same three-bar curve. It may be remarked that the intersection of the lines BB_1 and CC_1 is the axis of the instantaneous rotation of the triangle OB_1C_1, so that, joining this intersection with the point O, we have the normal at O to the locus; and similarly for the other two triangles. It of course follows that the intersections of BB_1 and CC_1, of CC_2 and AA_2, and of AA_3 and BB_3, lie on a line through O, viz., this line is the normal at O.

The result depends on the following theorem: viz., starting with the similar triangles OB_1C_1, A_2OC_2, A_3B_3O, say, the angles of these are A, B, C, so that the sides are

k_1 (sin A, sin B, sin C), k_2 (sin A, sin B, sin C), k_3 (sin A, sin B, sin C);

then it follows that the sides of the triangle ABC are

$$k \text{ (sin A, sin B, sin C)},$$

the value of k being given by the equation

$$k^2 = k_1^2 + k_2^2 + k_3^2 + 2k_2k_3 \cos (X-A) + 2k_3k_1 \cos (Y-B) + 2k_1k_3 \cos (Z-C),$$

where X, Y, Z denote the angles A_2OA_3, B_3OB_1, C_1OC_2 respectively: whence, since $A+B+C = \pi$, we have also $X+Y+Z = \pi$. If therefore the angles X, Y, Z vary in any manner subject to this last relation and to the equation $k^2 = \text{const.}$, the triangle ABC will be constant in magnitude.

There is no difficulty in proving the theorem. Writing $OC = \tau$, and $OB = \sigma$, also $\angle COC_1 = \psi$, and $\angle BOB_1 = \phi$, we have

$$\tau^2 = b_1^2 + a_2^2 + 2b_1a_2 \cos Z, \quad \frac{\sin \psi}{a_2} = \frac{\sin Z}{\tau}, \quad \cos \psi = \frac{b_1 + a_2 \cos Z}{\tau},$$

$$\sigma^2 = c_1^2 + a_3^2 + 2c_1a_3 \cos Y, \quad \frac{\sin \phi}{a_3} = \frac{\sin Y}{\sigma}, \quad \cos \phi = \frac{c_1 + a_3 \cos Y}{\sigma};$$

and then

$$\begin{aligned} a^2 &= \tau^2 + \sigma^2 - 2\tau\sigma \cos (A+\psi+\phi) \\ &= \tau^2 + \sigma^2 - 2\tau\sigma \cos A \cos \phi \cos \psi + 2\tau\sigma \sin A \sin \phi \sin \psi \\ &\qquad + 2\tau\sigma \sin A \sin \psi \cos \phi + 2\tau\sigma \sin A \cos \psi \sin \phi \\ &= b_1^2 + c_1^2 + a_2^2 + a_3^2 + 2b_1a_2 \cos Z + 2c_1a_3 \cos Y \\ &\quad - 2 \cos A \, (b_1 + a_2 \cos Z)(c_1 + a_3 \cos Y) \\ &\quad + 2 \cos A \, . \, a_2a_3 \sin Y \sin Z \\ &\quad + 2 \sin A \sin Z \, . \, a_2 (c_1 + a_3 \cos Y) \\ &\quad + 2 \sin A \sin Y \, . \, a_3 (b_1 + a_2 \cos Z) \end{aligned}$$

$$= b_1^2 + c_1^2 - 2b_1c_1 \cos A + a_2^2 + a_3^2$$
$$+ 2a_2a_3 [-\cos A (-\sin Y \sin Z + \cos Y \cos Z)$$
$$+ \sin A (\sin Y \cos Z + \cos Y \sin Z)]$$
$$+ 2a_3 [(c_1 - b_1 \cos A) \cos Y + b_1 \sin A \sin Y],$$
$$+ 2a_2 [(b_1 - c_1 \cos A) \cos Z + c_1 \sin A \sin Z].$$

We have here $b_1^2 + c_1^2 - 2b_1c_1 \cos A = a_1^2$, the second line is $= -2a_2a_3 \cos (A+Y+Z)$, which by virtue of $Y+Z = \pi - X$ is $= 2a_2a_3 \cos (X-A)$, and in the third and fourth lines

$$c_1 - b_1 \cos A = a_1 \cos B, \quad b_1 \sin A = a_1 \sin B,$$
$$b_1 - c_1 \cos A = a_1 \cos C, \quad c_1 \sin A = a_1 \sin C;$$

whence these lines are $2a_3a_1 \cos (Y-B)$, $2a_1a_2 \cos (Z-C)$: and the equation is

$$a^2 = a_1^2 + a_2^2 + a_3^2$$
$$+ 2a_2a_3 \cos (X-A) + 2a_3a_1 \cos (Y-B) + 2a_1a_2 \cos (Z-C),$$

which, putting therein for a_1, a_2, a_3 the values $k_1 \sin A$, $k_2 \sin A$, $k_3 \sin A$, and assuming as above

$$k^2 = k_1^2 + k_2^2 + k_3^2$$
$$+ 2k_2k_3 \cos (X-A) + 2k_3k_1 \cos (Y-B) + 2k_1k_2 \cos (Z-C),$$

becomes $a^2 = k^2 \sin^2 A$, or say $a = k \sin A$; and similarly $b = k \sin B$, $c = k \sin C$, that is, $(a, b, c) = k$ (sin A, sin B, sin C), the required theorem.

Before proceeding to find the equation of the curve, I insert, by way of lemma, the following investigation :—

Three triads (A, B, C), (F, G, H), (I, J, K) of points in a line, or of lines through a point, may be in cubic involution ; viz., representing A, B, &c. by the equations $x - ay = 0$, $x - by = 0$, &c., then this is the case when the cubic functions

$$(x-ay)(x-by)(x-cy), \ (x-fy)(x-gy)(x-hy), \ (x-iy)(x-jy)(x-ky)$$

are connected by a linear equation. Regarding I, J, K as given, the condition establishes between (A, B, C) and (F, G, H) two relations : viz., these are

$$(i-a)(i-b)(i-c) : (j-a)(j-b)(j-c) : (k-a)(k-b)(k-c)$$
$$= (i-f)(i-g)(i-h) : (j-f)(j-g)(j-h) : (k-f)(k-g)(k-h),$$

but if K be regarded as indeterminate, then the condition establishes only the single relation

$$(i-a)(i-b)(i-c) : (j-a)(j-b)(j-c)$$
$$= (i-f)(i-g)(i-h) : (j-f)(j-g)(j-h).$$

which relation, if $i = 0, j = \infty$, takes the form $abc = fgh$. When K is

thus indeterminate, we may say that the triads (A, B, C), (F, G, H) are in cubic involution with the duad I, J.

If A, B, &c. are points on a conic, then, considering the pencils obtained by joining these points with a point Θ on the conic, if the cubic involution exists for any particular position of Θ, it will exist for every position whatever of Θ; hence, considering triads of points on a conic, we may have a cubic involution between three triads, or between two triads and a duad, as above.

Taking $x = 0$, $y = 0$ for the equations of the tangents at the points I, J respectively, and $z = 0$ for the equation of the line joining these two points, the equation of the conic may be taken to be $xy - z^2 = 0$, and consequently the coordinates of any point A on the conic to be $x : y : z = \alpha : \frac{1}{\alpha} : 1$. It is then readily shown that, α, β, γ, f, g, h referring to the points A, B, C, F, G, H respectively, the condition for the cubic involution of (A, B, C), (F, G, H) with the duad (I, J) is $\alpha\beta\gamma = fgh$.

And we thence at once prove the theorem, that there exists a cubic curve $J_A I_B I_C FGH$, viz., a cubic curve passing through J, and having there the tangent JA, having at I a node with the tangents IB, IC to the two branches respectively, and passing through the points F, G, H; viz., that the triads (A, B, C), (F, G, H) being in cubic involution with (I, J) as above, there exists a cubic curve satisfying these $2+5+3, = 10$ conditions. In fact, the equation is

$$J_A I_B I_C FGH\,;\ \left(y - \frac{z}{\alpha}\right)(x-\beta z)(x-\gamma z)$$

$$+ \frac{z}{\alpha x}\{(x-\alpha z)(x-\beta z)(x-\gamma z) - (x-fz)(x-gz)(x-hz)\} = 0,$$

where observe that second term is an integral function $\frac{1}{\alpha} z^2 (-Mx + Nz)$, if for shortness,

$$M = \alpha+\beta+\gamma-f-g-h,$$

$$N = \beta\gamma+\gamma\alpha+\alpha\beta-gh-hf-fg.$$

In fact, the equations of the lines JA, IB, IC are $y - \frac{z}{\alpha} = 0$, $x-\beta z = 0$, $x-\gamma z = 0$, respectively, and we at once see that these lines are tangents at the points I, J respectively; moreover, at the point F, we have $x, y, z = f, \frac{1}{f}, 1$, and substituting these values the equation becomes

$$\left(\frac{1}{f} - \frac{1}{\alpha}\right)(f-\beta)(f-\gamma) + \frac{1}{\alpha f}(f-\alpha)(f-\beta)(f-\gamma) = 0;$$

viz., the equation is satisfied identically, or the curve passes through F, and similarly the curve passes through G and H.

In precisely the same manner there exists a cubic curve $I_A J_B J_C FGH$;

viz., this is

$$\mathrm{I_A J_B J_C FGH};\ (x-\alpha z)\left(y-\frac{z}{\beta}\right)\left(y-\frac{z}{\gamma}\right)$$

$$+\frac{\alpha z}{y}\left\{\left(y-\frac{z}{\alpha}\right)\left(y-\frac{z}{\beta}\right)\left(y-\frac{z}{\gamma}\right)-\left(y-\frac{z}{f}\right)\left(y-\frac{z}{g}\right)\left(y-\frac{z}{h}\right)\right\}=0,$$

where the second term is an integral function, $\alpha z^2(-\mathrm{M}'y+\mathrm{N}'z)$; if for shortness

$$\mathrm{M}'=\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}-\frac{1}{f}-\frac{1}{g}-\frac{1}{h},=\frac{1}{\alpha\beta\gamma}\mathrm{N},$$

$$\mathrm{N}'=\frac{1}{\beta\gamma}+\frac{1}{\gamma\alpha}+\frac{1}{\alpha\beta}-\frac{1}{gh}-\frac{1}{hf}-\frac{1}{fg}=\frac{1}{\alpha\beta\gamma}\mathrm{M},$$

in virtue of the relation $\alpha\beta\gamma=fgh$; so that the second term is in fact $=\frac{z^2}{\beta\gamma}(-\mathrm{C}x+\mathrm{B}z)$.

Writing for shortness $\mathrm{J_A}$, $\mathrm{I_A}$ to denote these two cubics respectively, we have four other like cubics, $\mathrm{J_B}(=\mathrm{J_B I_C I_A FGH})$, $\mathrm{I_B}(=\mathrm{J_B J_C J_A FGH})$, $(\mathrm{J_C}=\mathrm{J_C I_A I_B FGH})$, and $\mathrm{I_C}(=\mathrm{I_C J_A J_B FGH})$; the equations being

$$\mathrm{J_A};\ \left(y-\frac{z}{\alpha}\right)(x-\beta z)(x-\gamma z)+\frac{z^2}{\alpha}(-\mathrm{M}x+\mathrm{N}z)=0,$$

$$\mathrm{J_B};\ \left(y-\frac{z}{\beta}\right)(x-\gamma z)(x-\alpha z)+\frac{z^2}{\beta}(-\mathrm{M}x+\mathrm{N}z)=0,$$

$$\mathrm{J_C};\ \left(y-\frac{z}{\gamma}\right)(x-\alpha z)(x-\beta z)+\frac{z^2}{\gamma}(-\mathrm{M}x+\mathrm{N}z)=0,$$

$$\mathrm{I_A};\ (x-\alpha z)\left(y-\frac{z}{\beta}\right)\left(y-\frac{z}{\gamma}\right)+\frac{z^2}{\beta\gamma}(-\mathrm{N}y+\mathrm{M}z)=0,$$

$$\mathrm{I_B};\ (x-\beta z)\left(y-\frac{z}{\gamma}\right)\left(y-\frac{z}{\alpha}\right)+\frac{z^2}{\gamma\alpha}(-\mathrm{N}y+\mathrm{M}z)=0,$$

$$\mathrm{I_C};\ (x-\gamma z)\left(y-\frac{z}{\alpha}\right)\left(y-\frac{z}{\beta}\right)+\frac{z^2}{\alpha\beta}(-\mathrm{N}y+\mathrm{M}z)=0.$$

We require the differences of the products $\mathrm{J_A I_A}$, $\mathrm{J_B I_B}$, $\mathrm{J_C I_C}$. We find

$$\mathrm{I_B J_B}=(x-\alpha z)(x-\beta z)(x-\gamma z)\left(y-\frac{z}{\alpha}\right)\left(y-\frac{z}{\beta}\right)\left(y-\frac{z}{\gamma}\right)$$

$$+\frac{z^4}{\alpha\beta\gamma}(-\mathrm{M}x+\mathrm{N}z)(-\mathrm{N}y+\mathrm{M}z)\ (=\Omega \text{ suppose}).$$

$$+\frac{z^2}{\gamma\alpha}(-\mathrm{N}y+\mathrm{M}z)\left(y-\frac{z}{\beta}\right)(x-\gamma z)(x-\alpha z)$$

$$+\frac{z^2}{\beta}(-\mathrm{M}x+\mathrm{N}z)(x-\beta z)\left(y-\frac{z}{\gamma}\right)\left(y-\frac{z}{\alpha}\right).$$

and similarly

$$I_C J_C = \Omega + \frac{z^2}{\alpha\beta}(-My+Nz)\left(y-\frac{z}{\gamma}\right)(x-\alpha z)(x-\beta z)$$
$$+\frac{z^2}{\gamma}(-Mx+Nz)(x-\gamma z)\left(y-\frac{z}{\gamma}\right)\left(y-\frac{z}{\beta}\right).$$

We have thence

$$I_B J_B - I_C J_C = z^2\left\{\frac{1}{\alpha}(x-\alpha z)(-Ny+Mz)-\left(y-\frac{z}{\alpha}\right)(-Mx+Nz)\right\}$$
$$\times\left\{\frac{1}{\gamma}\left(y-\frac{z}{\beta}\right)(x-\gamma z)-\frac{1}{\beta}\left(y-\frac{z}{\gamma}\right)(x-\beta z)\right\},$$

and the factors in $\{\}$ are respectively

$$=\left(\frac{1}{\gamma}-\frac{1}{\beta}\right)(xy-z^2) \text{ and } \left(M-\frac{N}{\alpha}\right)(xy-z^2),$$

so that we have

$$I_B J_B - I_C J_C = \left(\frac{1}{\beta}-\frac{1}{\gamma}\right)\left(\frac{N}{\alpha}-M\right)z^2(xy-z^2)^2.$$

The constant factor $\left(\frac{1}{\beta}-\frac{1}{\gamma}\right)\left(\frac{N}{\alpha}-M\right)$ is

$$=\left(\frac{N}{\alpha\beta}+\frac{M}{\gamma}\right)-\left(\frac{N}{\gamma\alpha}+\frac{M}{\beta}\right), = P_3-P_2,$$

if P_1, P_2, P_3 denote respectively the functions

$$\frac{N}{\beta\gamma}+\frac{M}{\alpha}, \quad \frac{N}{\gamma\alpha}+\frac{M}{\beta}, \quad \frac{N}{\alpha\beta}+\frac{M}{\gamma},$$

and, attending to the equation $\alpha\beta\gamma = fgh$, it appears that we have

$$P_1 = \frac{1}{\alpha}(\alpha+\beta+\gamma-f-g-h)+\alpha\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}-\frac{1}{f}-\frac{1}{g}-\frac{1}{h}\right),$$

with the like values for P_1 and P_2.

We have thus

$$I_B J_B - I_C J_C = -(P_2-P_3)z^2(xy-z^2)^2,$$

and similarly

$$I_C J_C - I_A J_A = -(P_3-P_1)z^2(xy-z^2)^2,$$
$$I_A J_A - I_B J_B = -(P_1-P_2)z^2(xy-z^2)^2.$$

Any function $I_A J_A + \lambda z^2(xy-z^2)^2$, where λ is arbitrary, can of course be expressed in the form $I_A J_A + (\theta+P_1)z^2(xy-z^2)^2$, where θ is arbitrary, and therefore in the three equivalent forms

$$I_A J_A + (\theta+P_1)z^2(xy-z^2)^2,$$
$$I_B J_B + (\theta+P_2)z^2(xy-z^2)^2,$$
$$I_C J_C + (\theta+P_3)z^2(xy-z^2)^2.$$

We have $z=0$, the line IJ, and $xy-z^2=0$ the conic IJABCFGH.

The equation $I_A J_A + \lambda z^2 (xy - z^2)^2 = 0$ may thus be written in the more complete form

$$I_A J_B J_C FGH \,.\, J_A I_B I_C FGH + \lambda (IJ)^2 (IJABCFGH)^2 = 0,$$

and we hence see that it is the equation of a sextic curve, having a triple point at I, the tangents there being IA, IB, IC; having a triple point at J, the tangents there being JA, JB, JC, and having a node (double point) at each of the points F, G, H. There are thus in all $(6+3)+(6+3)+3+3+3, = 27$ conditions, and these would in general be sufficient to determine the sextic. The data are, however, related in a special manner; viz., regarding the points I, J, F, G, H as arbitrary, the lines IA, IB, IC, JA, JB, JC are not arbitrary, but satisfy the conditions that A, B are arbitrary points, and C a determinate point, on the conic IJABC. And the foregoing result shows that, this being so, there exists a sextic satisfying the foregoing conditions, but containing in its equation an arbitrary constant λ or θ, and that the equation may be presented under the three forms

$$I_A J_B J_C FGH \,.\, J_A I_B I_C \, FGH + (\theta + P_1) (IJ)^2 (IJABCFGH)^2 = 0, \text{ \&c.},$$

corresponding to the partitions A, BC; B, CA; C, AB of the three points A, B, C.

In the case where I, J are the circular points at infinity, the conic IJABCFGH is a circle containing upon it the six points A, B, C, F, G, H; and the condition of the cubic involution of the triads (A, B, C) and (F, G, H) with the points (I, J) is easily seen to be equivalent to the following relation, viz., the sum of the distances (measured along the circle from any fixed point of the circumference) of the three points A, B, C is equal to the sum of the distances of the three points F, G, H.

The sextic is a tricircular sextic having the three points A, B, C for foci, and having three nodes F, G, H, on the circle ABC, two of them being arbitrary points, and the third of them a determinate point on this circle. And it appears that there exists a sextic satisfying the foregoing conditions, and containing in its equation an arbitrary parameter.

I proceed to find the equation of the curve.

Consider the curve (see Fig. 1) as generated by the point O, the vertex of the triangle OC_1B_1, connected by the bars C_1C and B_1B with the fixed points C and B respectively; and suppose, as before, $CB = a$, $C_1C = a_2$, $B_1b = a_3$, $B_1C_1 = a_1$, $OC_1 = b_1$, $OB_1 = c_1$; and draw as in the figure the parallelograms C_1CC_2O and B_1BB_3O; then O may be considered as the intersection of a circle, centre C_2 and radius C_2O, with a circle, centre B_3 and radius B_3O. Take $\angle C_2CB = \theta$, $\angle B_3BC = \phi$, the lines CC_2, BB_3 are parallel to OC_1, OB_1 respectively, and consequently $\theta + \phi = \pi - A$, a relation between the two variable angles θ, φ.

Taking the origin at C and the axis of x along the line CB, that of y being at right angles to it, the coordinates of C_2 are $(b_1 \cos\theta,\ b_1 \sin\theta)$, and those of B_3 are $(a-c_1\cos\phi,\ c_1 \sin\phi)$; the equations of the circles thus are

$$(x-b_1\cos\theta)^2 + (y-b_1\sin\theta)^2 = a_2^2,$$
$$(x-a+c_1\cos\phi)^2 + (y-c_1\sin\phi)^2 = a_3^2;$$

whence

$$+2b_1x\cos\theta + 2b_1y\sin\theta = x^2+y^2+b_1^2-a_2^2,$$
$$-2c_1(x-a)\cos\phi + 2c_1y\sin\phi = (x-a)^2+y^2+c_1^2-a_3^2,$$

which equations, writing therein for θ its value $= \pi - A - \phi$, and eliminating the single parameter ϕ, give the equation of the curve.

We in fact have

$$-2b_1x\cos(A+\phi) + 2b_1y\sin(A+\phi) = x^2+y^2+b_1^2-a_2^2,$$
$$-2c_1(x-a)\cos\phi \quad + 2c_1y\sin\phi \quad = (x-a)^2+y^2+c_1^2-a_3^2;$$

or say these are

$$P\cos\phi + Q\sin\phi = R,$$
$$P'\cos\phi + Q'\sin\phi = R',$$

where

$$P = -2b_1x\cos A + 2b_1y\sin A, \quad P' = -2c_1(x-a),$$
$$Q = 2b_1x\sin A + 2b_1y\cos A, \quad Q' = 2c_1y,$$
$$R = x^2+y^2+b_1^2-a_2^2, \quad R' = (x-a)^2+y^2+c_1^2-a_3^2;$$

and the equations give therefore

$$\cos\phi : \sin\phi : -1 = QR'-Q'R : RP'-R'P : PQ'-P'Q,$$

whence

$$(QR'-Q'R)^2 + (RP'-R'P)^2 = (PQ'-P'Q)^2;$$

and it hence follows that the nodes are the common intersections of the three curves

$$QR'-Q'R = 0,\ RP'-R'P = 0,\ PQ'-P'Q = 0.$$

We have, retaining R, R′ to denote their values,

$$QR'-Q'R = -2[(Rc_1-R'b_1\cos A)\,y - R'b_1\sin A\,.\,x],$$
$$RP'-R'P = -2[(Rc_1-R'b_1\cos A)(x-a) + R'b_1\sin A\,(y-a\cot A)],$$
$$PQ'-P'Q = -4b_1c_1[x(x-a)+y(y-a\cot A)].$$

Observing that $R=0$, $R'=0$ are circles; the equation $QR'-Q'R=0$ is a circular cubic through the point $x=0$, $y=0$; the equation $RP'-R'P=0$, a circular cubic through the point $x=a$, $y=a\cot A$; and the equation $PQ'-P'Q=0$, a circle through these two points (and also the points $x=0$, $y=a\cot A$; $x=a$, $y=0$). Hence the first and third curves intersect in the point $(x=0, y=0)$, the circular points at infinity, and three other points which are the nodes; viz., the curve has three nodes, say these are F, G, H. The second and third curves intersect in the point $x=0$, $y=a\cot A$, the circular points at infinity, and the three nodes. As regards the first and second curves, it

is readily shown that these touch at the circular points at infinity; viz., they intersect in these points each twice, in the two finite intersections of the circles $R = 0$, $R' = 0$, and in the three nodes.

The three nodes F, G, H thus lie in the circle

$$x(x-a)+y(y-a\cot A) = 0,$$

which passes through the points $(x = 0, y = 0)$ and $(x = a, y = 0)$, that is, the points C and B. Assuming $b = \frac{a \sin A}{\sin B}$, the circle also passes through the point $x = b\cos C$, $y = b \sin C$, that is, the point A of the figure. Thus the three nodes F, G, H lie in the circle circumscribed about the triangle ABC.

Writing, for greater convenience,

$$R = x^2+y^2-e^2,\ R' = x^2+y^2-2ax-f^2,$$

the nodes F, G, H lie on the two curves

$$c_1 y(x^2+y^2-e^2)-b_1 \sin A (x+y\cot A)(x^2+y^2-2ax-f^2) = 0,$$
$$x^2+y^2 = a(x+y\cot A).$$

The first of these is

$$[c_1 y-b_1 \sin A (x+y\cot A)](x^2+y^2)$$
$$+[b_1 \sin A (x+y\cot A) f^2-c_1 e^2 y]$$
$$+2ab_1 \sin A (x+y\cot A) x = 0.$$

We may combine these equations so as to obtain the equation of the triad of lines CF, CG, CH; viz., multiplying the second and third terms of the first equation by $\frac{(x^2+y^2)^2}{a^2(x+y\cot A)^2}$ and $\frac{x^2+y^2}{a(x+y\cot A)}$ (each $= 1$ in virtue of the second equation), the equation becomes divisible by x^2+y^2; and, throwing this out, the equation is

$$c_1 y-b_1 \sin A (x+y\cot A)$$
$$+[b_1 \sin A (x+y\cot A) f^2-c_1 e^2 y]\frac{x^2+y^2}{a^2(x+y\cot A)^2}$$
$$+2b_1 x \sin A = 0,$$

where the first and third terms together are $= (c_1-b_1\cos A)y+b_1 \sin A . x$, viz., this is $= a_1 \sin B (x+y\cot B)$. Hence, writing also in the second term $a_1 \sin B$ for $b_1 \sin A$, the equation is

$$(x+y\cot A)^2 (x+y\cot B)$$
$$+\frac{1}{a^2}\left((x+y\cot A) f^2-\frac{c_1 e^2}{a_1 \sin B} y\right)(x^2+y^2) = 0;$$

or say this is

$$(x\sin A+y\cos A)^2 (x\sin B+y\cos B)$$
$$+\frac{\sin A \sin B}{a^2}\left\{(x\sin A+y\cos A) f^2-\frac{c_1 e^2}{b_1} y\right\}(x^2+y^2) = 0;$$

viz., there is a term in x^2+y^2, and another term

$$(x\sin A+y\cos A)^2 (x\sin B+y\cos B).$$

Suppose for a moment the angles FBC, GBC, HBC are called F, G, H, then the function on the left hand must be

$$= \mathrm{M}\,(x\sin \mathrm{F} - y\cos \mathrm{F})\,(x\sin \mathrm{G} - y\cos \mathrm{G})\,(x\sin \mathrm{H} - y\cos \mathrm{H}).$$

Writing in the identity $x = iy$, we have

$$(\cos \mathrm{A} + i\sin \mathrm{A})^2\,(\cos \mathrm{B} + i\sin \mathrm{B})$$
$$= -\mathrm{M}\,(\cos \mathrm{F} - i\sin \mathrm{F})\,(\cos \mathrm{G} - i\sin \mathrm{G})\,(\cos \mathrm{H} - i\sin \mathrm{H});$$

and similarly, writing $x = -iy$, we have the like equation with $-i$ instead of $+i$; whence, dividing the two equations and taking the logarithms,

$$4\mathrm{A} + 2\mathrm{B} = 2m\pi - \mathrm{F} - \mathrm{G} - \mathrm{H}.$$

which leads as before to the relation $\mathrm{A}' + \mathrm{B}' + \mathrm{C}' = \mathrm{F}' + \mathrm{G}' + \mathrm{H}'$

In completion of the investigation, observe that M is determinately $+1$ or -1, and that

$$\frac{\sin \mathrm{A}\sin \mathrm{B}}{a^2}\left\{(x\sin \mathrm{A} + y\cos \mathrm{A})\,f^2 - \frac{c_1}{b_1}e^2 y\right\}$$

is the linear factor of

$$\mathrm{M}\,(x\sin \mathrm{F} - y\cos \mathrm{F})\,(x\sin \mathrm{G} - y\cos \mathrm{G})\,(x\sin \mathrm{H} - y\cos \mathrm{H})$$
$$-(x\sin \mathrm{A} + y\cos \mathrm{A})^2\,(x\sin \mathrm{B} + y\cos \mathrm{B}),$$

which remains after throwing out the factor $x^2 + y$. Calling this linear factor $px + qy$, we have

$$\frac{a^2 p}{\sin \mathrm{A}\sin \mathrm{B}} = f^2\sin \mathrm{A}, \quad \frac{a^2 q}{\sin \mathrm{A}\sin \mathrm{B}} = f^2\cos \mathrm{A} - \frac{c_1}{b_1}e^2;$$

or, as this last equation may be written,

$$\frac{a^2 q}{\sin \mathrm{A}\sin \mathrm{B}} = f^2\cos \mathrm{A} - \frac{\sin \mathrm{C}}{\sin \mathrm{B}}e^2.$$

Hence, writing $a = k\sin \mathrm{A}$, we have

$$f^2 = \frac{k^2 p}{\sin \mathrm{B}}, \quad e^2 = \frac{k^2}{\sin \mathrm{C}}(p\cos \mathrm{A} - q\sin \mathrm{A});$$

or, substituting for f^2, e^2 their values,

$$-k_1^2\sin^2 \mathrm{C} + k_3^2\sin^2 \mathrm{A} = \frac{k^2 p}{\sin \mathrm{B}} + k^2\sin^2 \mathrm{A},$$

$$-k_1^2\sin^2 \mathrm{B} + k_2^2\sin^2 \mathrm{A} = \frac{k^2}{\sin \mathrm{C}}(p\cos \mathrm{A} - q\sin \mathrm{A}),$$

or, what is the same thing,

$$-\frac{k_1^2}{\sin^2 \mathrm{A}} + \frac{k_3^2}{\sin^2 \mathrm{C}} = \frac{k^2}{\sin^2 \mathrm{A}\sin \mathrm{B}\sin^2 \mathrm{C}}(p + \sin^2 \mathrm{A}\sin \mathrm{B}),$$
$$= \frac{k^2\sin \mathrm{F}\sin \mathrm{G}\sin \mathrm{H}}{\sin \mathrm{A}\sin \mathrm{C}\,.\,\sin \mathrm{A}\sin \mathrm{B}\sin \mathrm{C}},$$

$$-\frac{k_1^2}{\sin^2 \mathrm{A}} + \frac{k_2^2}{\sin^2 \mathrm{B}} = \frac{k^2}{\sin^2 \mathrm{A}\sin^2 \mathrm{B}\sin \mathrm{C}}(p\cos \mathrm{A} - q\sin \mathrm{A}),$$
$$= \frac{k^2\sin(\mathrm{A}-\mathrm{F})\sin(\mathrm{A}-\mathrm{G})\sin(\mathrm{A}-\mathrm{H})}{\sin \mathrm{A}\sin \mathrm{B}\,.\,\sin \mathrm{A}\sin \mathrm{B}\sin \mathrm{C}},$$

which are the relations connecting k_1, k_2, k_3, when the foci and nodes are given.

It is to be remarked that if, for instance, $F = 0$ and $G = A$, then $k_1 : k_2 : k_3 = \sin A : \sin B : \sin C$; the nodes in this case coincide with the foci. A simple example is when $A = B = C$; the three triangles are here equal equilateral triangles. The general equations show that, if l_1, l_2, l_3 are values of k_1, k_2, k_3 belonging to a given set of nodes and foci, then the values $k_1^2 = l_1^2 + u \sin^2 A$, $k_2^2 = l_2^2 + u \sin^2 B$, $k_3^3 + u \sin^2 C$ (where u is arbitrary) will belong to the same set of nodes and foci.

I write the equation of the curve in the form

$$\{(QR' - Q'R) + i(RP' - R'P)\}\{QR' - Q'R - i(RP' - RP)\} - (PQ' - P'Q)^2 = 0,$$

where

$$(QR' - Q'R) + i(RP' - R'P)$$

$$= (Rc_1 - R'b_1 \cos A)\, i(x - a - iy) - R'b_1 \sin A\,\{x - i(y - a \cot A)\},$$

Calling I, J the circular points $(\infty, x + iy = 0)$ and $(\infty, x - iy = 0)$, this is a nodal circular cubic having I for an ordinary point, but J for a node. Moreover, one of the tangents at J is the line $x - iy = 0$, that is, the line JB; in fact, writing as before

$$R = x^2 + y^2 - e^2, \quad R' = x^2 + y^2 - 2ax - f^2,$$

then, when $x - iy = 0$, we have $R = -e^2$, $R' = -2ax - f^2$, and the equation becomes

$$\{-ce^2 + b_1 \cos A\,(2ax + f^2)\}(-ia) + b_1 \sin A\,(2ax + f^2)\,(ia \cot A) = 0;$$

viz., the term in x here disappears, or the three intersections are at infinity. The other tangent at J is the line $x - a - iy = 0$, that is, the line JC; in fact, when $x - a - iy = 0$, that is, $y = -i(x - a)$, we have $R = 2ax - a^2 - e^2$, $R' = -a^2 - f^2$, and the equation becomes

$$\{c_1(2ax - a^2 - e^2) + b_1 \cos A\,(a^2 + f^2)\} \,.\, 0 + b_1 \sin A\,(a^2 + f^2) \,.\, a(1 + i \cot A) = 0,$$

viz., the three intersections are here at infinity. The tangent at I is the line $x - b \cos C + i(y - b \sin C) = 0$, that is, the line IA; in fact, writing this in the form

$$y = ix - ib(\cos C + i \sin C) = ix - ib\gamma,$$

(if for a moment $\cos C + i \sin C = \gamma$, and similarly $\cos A + i \sin A = \alpha$, $\cos B + i \sin B = \beta$); then, y having this value, we find

$$R = 2bx\gamma - b^2\gamma^2 - e^2, \quad R' = 2(b\gamma - a)\,x - b^2\gamma^2 - f^2,$$

$$= -\frac{2c}{\beta}\,x - b^2\gamma^2 - f^2;$$

and the equation becomes

$$\left\{\begin{matrix} C_1(2b\gamma x - b^2\gamma^2 - e^2) \\ -b_1\cos A\left(-\frac{2c}{\beta}x - b^2\gamma^2 - e^2\right)\end{matrix}\right\} i(2x - a - b\gamma)$$

$$-l_1\sin A\left(-\frac{2c}{\beta}x - b^2\gamma^2 - e^2\right)(2x + ia\cot A - b\gamma) = 0,$$

viz., the coefficient of x^2 is here

$$2i\left(bc_1\gamma + \frac{b_1c}{\beta}\cdot\frac{1}{a}\right),$$

or, since $b_1c = bc_1$, this is

$$= 2ibc_1\left(\gamma + \frac{1}{a\beta}\right), = 0,$$

in virtue of the relation $A+B+C = \pi$, giving $a\beta\gamma = -1$: hence there is only one finite intersection, or the line IA is a tangent.

The cubic in question

$$QR' - Q'R + i(RP' - R'P) = 0$$

is thus a nodal circular cubic which it is convenient to represent in the form

$$(I_AJ_BJ_CFGH) = 0;$$

viz., this a cubic, through I with the tangent IA, having J as a node with the tangents JB, JC, and through the points F, G, H. Observe that, if F, G, H were arbitrary, this would be $2+5+3, =10$ conditions. The before mentioned relation is in fact the condition in order to the existence of the cubic.

Similarly the cubic

$$QR' - Q'R - i(RP' - R'P) = 0$$

is the cubic

$$(J_AI_BI_CFGH) = 0.$$

The circle $PQ' - P'Q = 0$ is the quadric through I, J, A, B, C, F, G, H; or it may in like manner be written $(IJABCFGH) = 0$; and we may write $(IJ) = 0$, as the equation of the line infinity. The functions denoted as above contain implicitly constant multipliers which give, in the equation of the three-bar curve, one arbitrary parameter — and the equation thus is

$$(I_AJ_BJ_CFGH)(J_AI_BI_CFGH) - \theta(IJ)^2(IJABCFGH)^2 = 0,$$

a form which puts in evidence that I, J are triple points having the tangents IA, IB, IC, and JA, JB, JC respectively (whence also A, B, C are foci), and that F, G, H are nodes; viz., the result is as follows:—

Taking A, B, C, F, G, H points in a circle, such that, Sum of the distances (angular distances from a fixed point in the circumference) of A, B, C is equal to the sum of the distances of F, G, H, then there

exist the cubics $(I_A J_B J_C F G H) = 0$, $(J_A I_B I_C F G H) = 0$, and the sextic is as above.

Writing for shortness

$$(I_A J_B J_C F G H) = I_A, \quad (J_A I_B I_C F G H) = J_A,$$

then the above form is clearly one of three equivalent forms

$$\begin{aligned} U &= I_A J_A - \theta_1 \Omega^2 \\ &= I_B J_B - \theta_2 \Omega^2 \\ &= I_C J_C - \theta_3 \Omega^2, \end{aligned}$$

and this implies an identical linear relation between the functions $I_A J_A$, $I_B J_B$, $I_C J_C$; whence also U and Ω^2 are each of them a linear function of any two of these quantities.

I originally obtained the equation of the curve in a form which, though far less valuable than the preceding one, is nevertheless worth preserving; viz., the equation

$$(QR'-Q'R)^2 + (RP'-R'P)^2 = (PQ'-P'Q)^2$$

may be written

$$(R^2-P^2-Q^2)(R'^2-P'^2-Q'^2) - (RR'-PP'-QQ')^2 = 0,$$

which equation, substituting therein for P, Q, R, P', Q', R' their values, gives the form in question.

Proceeding to the reduction

$$\begin{aligned} R^2-P^2-Q^2 &= (x^2+y^2+b_1^2-a_2^2)^2 - 4b_1^2(x^2+y^2), \\ &= (x^2+y^2)^2 - 2(b_1^2+a_2^2)(x^2+y^2) + (b_1^2-a_2^2)^2, \\ &= (x^2+y^2-\overline{b_1+a_2}^2)(x^2+y^2-\overline{b_1-a_2}^2); \\ R'^2-P'^2-Q'^2 &= (\overline{x-a}^2+y^2+c_1^2-a_3^2)^2 - 4c_1^2(\overline{x-a}^2+y^2), \\ &= (\overline{x-a}^2+y^2)^2 - 2(c_1^2+a_3^2)(\overline{x-a}^2+y^2) + (c_1^2-a_3^2)^2, \\ &= (\overline{x-a}^2+y^2-\overline{c_1+a_3}^2)(\overline{x-a}^2+y^2-\overline{c_1-a_3}^2). \end{aligned}$$

But the reduction of $RR'-PP'-QQ'$ is somewhat longer. We have

$$\begin{aligned} RR'-PP'-QQ' = &(x^2+y^2+b_1^2-a_2^2)(\overline{x-a}^2+y^2+c_1^2-a_3^2) \\ &-4b_1c_1\cos A\,(x\,\overline{x-a}+y^2) - 4b_1c_1\sin A\,.\,ay, \end{aligned}$$

and here

$$2b_1c_1\cos A = b_1^2+c_1^2-a_1^2, \quad 2x(x-a)+2y^2 = x^2+y^2+\overline{x-a}^2+y^2-a^2,$$

also $b_1c_1\sin A = a_1p_1$, if p_1 be the perpendicular distance of O from the base B_1C_1. Hence the second line is

$$-(b_1^2+c_1^2-a_1^2)(x^2+y^2+\overline{x-a}^2+y^2-a^2) - 4aa_1p_1y,$$

and the whole is

$$\begin{aligned} = \quad & (x^2+y^2)\,(\overline{x-a}^2+y^2) \\ & +(x^2+y^2)\,(a_1^2-b_1^2-a_3^2) \\ & +(\overline{x-a}^2+y^2)\,(a_1^2-c_1^2-a_2^2) \\ & +(b_1^2-a_2^2)(c_1^2-a_3^2)+a^2(b_1^2+c_1^2-a_1^2)-4aa_1p_1y\,; \end{aligned}$$

whence, finally, we have

$$\begin{aligned} \mathrm{RR'-PP'-QQ'} = & (x^2+y^2+a_1^2-a_2^2-c_1^2)\,(\overline{x-a}^2+y^2+a_1^2-a_3^2-b_1^2) \\ & +(a^2+a_1^2-a_2^2-a_3^2)\,(b_1^2+c_1^2-a_1^2)-4aa_1p_1y\,; \end{aligned}$$

and hence the equation of the curve is

$$\begin{aligned} & (x^2+y^2-\overline{b_1+a_2}^2)\,(x^2+y^2-\overline{b_1-a_2}^2)\,(\overline{x-a}^2+y^2-\overline{c_1+a_3}^2) \\ & \qquad\qquad \times(\overline{x-a}^2+y^2-\overline{c_1-a_3}^2) \\ & -\{(x^2+y^2+a_1^2-a_2^2-c_1^2)\,(\overline{x-a}^2+y^2+a_1^2-a_3^2-b_1^2) \\ & \qquad +(a^2+a_1^2-a_2^2-a_3^2)\,(b_1^2+c_1^2-a_1^2)-4aa_1p_1y\}^2=0, \end{aligned}$$

where p_1 is given in terms of the constants a_1, b_1, c_1 by the equation

$$2a_1p_1=\sqrt{2b_1^2c_1^2+2c_1^2a_1^2+2a_1^2b_1^2-a_1^4-b_1^4-c_1^4}\,.$$

There are in the equation two terms, $(x^2+y^2)^2$, $(\overline{x-a}^2+y^2)^2$, which destroy each other, and the remaining terms are of the order 6 at most. Hence the curve is a sextic; and it is, moreover, readily seen that the curve is tricircular. Assuming this, it appears at once that the lines $x+iy=0$, $x-iy=0$ are tangents to the curve at the two circular points at infinity. In fact, assuming either of these equations, we have $x^2+y^2=0$, and the equation becomes

$$\begin{aligned} & (b_1^2-a_2^2)\,(-2ax+a^2-\overline{c_1+a_3}^2)\,(-2ax+a^2-\overline{c_1-a_3}^2) \\ & \quad -\{(a_1^2-a_2^2-c_1^2)\,(-2ax+a^2+a_1^2-a_3^2-b_1^2) \\ & \qquad +(a^2+a_1^2-a_2^2-a_3^2)\,(b_1^2+c_1^2-a_1^2)-4aa_1p_1y\}^2=0, \end{aligned}$$

a quadric equation. Hence there are on each of the two lines only two finite intersections, or the number of intersections at infinity is $=4$; viz., the line is a tangent to one of the branches at the triple point. Similarly, the lines $x-a+iy=0$, $x-a-iy=0$ are tangents. Thus the points C and B are foci. It might with somewhat more difficulty be shown from the equation that the point $x=b\cos\mathrm{C}$, $y=b\sin\mathrm{C}$ $\left(\text{where, as before, } b=\dfrac{a\sin\mathrm{A}}{\sin\mathrm{B}}\right)$, viz., the point A of the figure, is a focus; but I have not verified this directly. It clearly follows, if we generate the curve by means of the triangle $\mathrm{OA_2C_2}$ and the fixed

points C, A. Hence A, B, C are a triad of foci, and the theorem as to the nodes is that these lie on the circle drawn through the three foci A, B, C.

I prove in a somewhat different manner, for the sake of the further theory which arises, the theorem of the triple generation; for this purpose, constructing the foregoing Figure 2 by means of the three triangles OB_1C_1, OC_2A_2, OA_3B_3, but without assuming anything as to the form or position of the triangle ABC, I draw through O a line Ox, the position of which is in the first instance arbitrary, say its inclination to OC_2 is $= v$; and drawing Oy at right angles to Ox, I proceed, in regard to these axes, to find the coordinates of the points C, B. We have,

for C, $x = a_2 \cos v + b_1 \cos (v+Z)$, $y = a_2 \sin v + b_1 \sin (v+Z)$;

for B, $x = c_1 \cos (v+A+Z) + a_3 \cos (v+A+Z+Y)$,

$y = c_1 \sin (v+A+Z) + a_3 \sin (v+A+Z+Y)$;

or, writing for $Y+Z$ the value $\pi - X$, so that

$$v+A+Z+Y = \pi + v + A - X,$$

the coordinates of B are

$$x = c_1 \cos (v+A+Z) - a_3 \cos (v+A-X),$$
$$y = c_1 \sin (v+A+Z) - a_3 \sin (v+A-X).$$

Taking the two values of y equal to each other, viz., writing to determine v the equation

$$a_2 \sin v + b_1 \sin (v+Z) - c_1 \sin (v+A+Z) + a_3 \sin (v+A-X) = 0,$$

we make the line Ox parallel to BC, so that, writing

$$x = a_2 \cos v + b_1 \cos (v+Z),$$
$$x - a = c_1 \cos (v+A+Z) - a_3 \cos (v+A-X),$$

we have

$$a = a_2 \cos v + b_1 \cos (v+Z) - c_1 \cos (v+A+Z) + a_3 \cos (v+A-X),$$

which determines the distance BC, $= a$. And moreover, writing

$$y = a_2 \sin v + b_1 \sin (v+Z),$$
$$= c_1 \sin (v+A+Z) - a_3 \sin (v+A-X),$$

we have y as the perpendicular distance of O from BC, and x and $(a-x)$ as the two parts into which BC is divided by the foot of this perpendicular. In the reduction of the formulæ we assume that the three triangles are similar; viz., we write

$$(a_1, b_1, c_1), \ (a_2, b_2, c_2), \ (a_3, b_3, c_3)$$
$$= k_1(\sin A, \sin B, \sin C), \ k_2(\sin A, \sin B, \sin C), \ k_3(\sin A, \sin B, \sin C);$$

and we use when required the relation $A+B+C = \pi$.

The equation for v becomes

$$k_1 \sin(v-C+Z) + k_2 \sin v + k_3 \sin(v+A-X) = 0,$$

which may be written $L \sin v - M \cos v = 0$,

where $$L = k_2 + k_1 \cos(Z-C) + k_3 \cos(X-A),$$
$$M = -k_1 \sin(Z-C) + k_3 \sin(X-A);$$

hence, putting

$$k^2 = k_1^2 + k_2^2 + k_3^2 + 2k_2k_3 \cos(X-A) + 2k_3k_1 \cos(Y-B) + 2k_1k_2 \cos(Z-C),$$

we have $L^2+M^2 = k^2$, that is, $\sqrt{L^2+M^2} = k$, and therefore

$$k \sin v = M, \quad k \cos v = L,$$

which gives the value of v; and then, after all reductions,

$$\begin{aligned} kx = {} & k_1^2 \sin B \cos C + k_2^2 \sin A + k_3^2 . 0 + k_2k_3 \sin A \cos(X-A) \\ & + k_3k_1[-\sin B \cos(Y+A)] \\ & + k_1k_2[\sin(B-A)\cos(Z+A) + 2\sin A \sin B \sin(Z+A)], \\ k(a-x) = {} & k_1^2 \sin C \cos B + k_2^2 . 0 + k_3^2 \sin A + k_2k_3 \sin A \cos(X-A) \\ & + k_3k_1[\sin(C-A)\cos(Y+A) + 2\sin A \sin C \sin(Y+A)] \\ & + k_1k_2[-\sin C \cos(Z+A)], \end{aligned}$$

and

$$ky = k_1^2 \sin B \sin C + k_2k_3 \sin A \sin(X-A) + k_3k_1 \sin B \sin(Y+A) + k_1k_2 \sin C \sin(Z+A).$$

The first and second equations give $ka = k^2 \sin A$, that is, $a = k \sin A$; and, similarly, $b = k \sin B$, $c = k \sin C$; viz., we have

$$(a, b, c) = k(\sin A, \sin B, \sin C),$$

or the triangle ABC is similar to the other three triangles, its magnitude being given by the foregoing equation for k^2. These are the properties which give the triple generation.

Changing the notation of the coordinates, and writing (x, y, z) for the perpendicular distances from O on the sides of the triangle ABC, we have, as above,

$$kx = k_1^2 \sin B \sin C + k_2k_3 \sin A \sin(X-A) + k_3k_1 \sin B \sin(Y+A) + k_1k_2 \sin C \sin(Z+A),$$

and therefore

$$ky = k_2^2 \sin B \sin C + k_2k_3 \sin A \sin(X+B) + k_3k_1 \sin B \sin(Y-B) + k_1k_2 \sin C \sin(Z+B),$$

$$kz = k_3^2 \sin C \sin A + k_2k_3 \sin A \sin(X+C) + k_3k_1 \sin B \sin(Y+C) + k_1k_2 \sin C \sin(Z-C),$$

values which give, as they should do,

$$x \sin A + y \sin B + z \sin C = k^2 \sin A \sin B \sin C.$$

Taking (x, y, z) as simply proportional (instead of equal) to the perpendicular distances, then (x, y, z) will be a system of trilinear coordinates in which the equation of the line-infinity is

$$x \sin A + y \sin B + z \sin C = 0;$$

and considering (x, y, z) as proportional to the foregoing values, and in these X, Y, Z as connected by the equation $X+Y+Z=\pi$, and the equation which determines k^2, the coordinates (x, y, z) are given as proportional to functions of a single parameter, so that the equations in effect determine the curve which is the locus of O.

But to determine the order, &c., the trigonometrical functions must be expressed algebraically; and this is done most readily by introducing instead of X, Y, Z the functions

$$\cos X + i \sin X, \quad \cos Y + i \sin Y, \quad \cos Z + i \sin Z, = \xi, \eta, \zeta;$$

and we may at the same time, in place of A, B, C, introduce the functions

$$\cos A + i \sin A, \quad \cos B + i \sin B, \quad \cos C + i \sin C, = \alpha, \beta, \gamma.$$

The relation $X+Y+Z=\pi$ gives $\xi\eta\zeta = -1$, and similarly $A+B+C=\pi$ gives $\alpha\beta\gamma = -1$.

We have

$$\cos(X-A) = \tfrac{1}{2}\left(\frac{\xi}{\alpha} + \frac{\alpha}{\xi}\right), \quad i \sin(X-A) = \tfrac{1}{2}\left(\frac{\xi}{\alpha} - \frac{\alpha}{\xi}\right), \quad \&c.;$$

the equation $k^2 = k_1^2 + \&c.$ becomes

$$k^2 = k_1^2 + k_2^2 + k_3^2 + k_2k_3\left(\frac{\xi}{\alpha} + \frac{\alpha}{\xi}\right) + k_3k_1\left(\frac{\eta}{\beta} + \frac{\beta}{\eta}\right) + k_1k_2\left(\frac{\zeta}{\gamma} + \frac{\gamma}{\zeta}\right),$$

or, as this may be written,

$$(-k^2 + k_1^2 + k_2^2 + k_3^2)$$
$$+ k_2k_3\left(\frac{\xi}{\alpha} - \alpha\eta\zeta\right) + k_3k_1\left(\frac{\eta}{\beta} - \beta\zeta\xi\right) + k_1k_2\left(\frac{\zeta}{\gamma} - \gamma\xi\eta\right) = 0.$$

Also the value of x is proportional to

$$k_1^2\left(\beta - \frac{1}{\beta}\right)\left(\gamma - \frac{1}{\gamma}\right) + k_2k_3\left(\alpha - \frac{1}{\alpha}\right)\left(\frac{\xi}{\alpha} - \frac{\alpha}{\xi}\right)$$
$$+ k_3k_1\left(\beta - \frac{1}{\beta}\right)\left(\alpha\eta - \frac{1}{\alpha\eta}\right) + k_1k_2\left(\gamma - \frac{1}{\gamma}\right)\left(\alpha\zeta - \frac{1}{\alpha\zeta}\right),$$

or, what is the same thing, to

$$k_1^2\left(\beta - \frac{1}{\beta}\right)\left(\gamma - \frac{1}{\gamma}\right) + k_2k_3\left(\alpha - \frac{1}{\alpha}\right)\left(\frac{\xi}{\alpha} + \alpha\eta\zeta\right)$$
$$+ k_3k_1\left(\beta - \frac{1}{\beta}\right)\left(\alpha\eta + \frac{\zeta\xi}{\alpha}\right) + k_1k_2\left(\gamma - \frac{1}{\gamma}\right)\left(\alpha\zeta + \frac{\xi\eta}{\alpha}\right);$$

with the like expressions as to the values of y and z. Introducing for homogeneity a quantity ω, viz., writing $\frac{y}{\omega}$, $\frac{\eta}{\omega}$, $\frac{\zeta}{\omega}$ in place of ξ, η, ζ, we have the parameters $(\xi, \eta, \zeta, \omega)$ connected by the homogeneous equations

$$\xi\eta\zeta + \omega^3 = 0,$$

$$(-k^2 + k_1^2 + k_2^2 + k_3^2)\,\omega^2$$

$$+ k_2k_3\left(\frac{\xi\omega}{a} - a\eta\zeta\right) + k_3k_1\left(\frac{\eta\omega}{\beta} - \beta\zeta\xi\right) + k_1k_2\left(\frac{\zeta\omega}{\gamma} - \gamma\xi\eta\right) = 0,$$

and the ratios of the coordinates are

$x : y : z$

$$= k_1^2\left(\beta - \frac{1}{\beta}\right)\left(\gamma - \frac{1}{\gamma}\right)\omega^2 + k_2k_3\left(a - \frac{1}{a}\right)\left(\frac{\xi\omega}{a} + a\eta\zeta\right)$$

$$+ k_3k_1\left(\beta - \frac{1}{\beta}\right)\left(a\eta\omega + \frac{\xi\zeta}{a}\right) + k_1k_2\left(\gamma - \frac{1}{\gamma}\right)\left(a\zeta\omega + \frac{\xi\eta}{a}\right)$$

$$: k_2^2\left(\gamma - \frac{1}{\gamma}\right)\left(a - \frac{1}{a}\right)\omega^2 + k_2k_3\left(a - \frac{1}{a}\right)\left(\beta\xi\omega + \frac{\eta\zeta}{\beta}\right)$$

$$+ k_3k_1\left(\beta - \frac{1}{\beta}\right)\left(\frac{\eta\omega}{\beta} + \beta\zeta\xi\right) + k_1k_2\left(\gamma - \frac{1}{\gamma}\right)\left(\beta\zeta\omega + \frac{\xi\eta}{\beta}\right)$$

$$: k_3^2\left(a - \frac{1}{a}\right)\left(\beta - \frac{1}{\beta}\right)\omega^2 + k_2k_3\left(a - \frac{1}{a}\right)\left(\gamma\xi\omega + \frac{\eta\zeta}{\gamma}\right)$$

$$+ k_3k_1\left(\beta - \frac{1}{\beta}\right)\left(\gamma\eta\omega + \frac{\zeta\xi}{\gamma}\right) + k_1\left(\gamma - \frac{1}{\gamma}\right)\left(\frac{\zeta\omega}{\gamma} + \gamma\xi\eta\right).$$

Suppose, for shortness, these are $x : y : z = \mathrm{P} : \mathrm{Q} : \mathrm{R}$. Observe that the form of the equations is $\xi\eta\zeta + \omega^3 = 0$, $\Omega = 0$, and $x : y : z = \mathrm{P} : \mathrm{Q} : \mathrm{R}$, where Ω and P, Q, R are each of them a quadric function of the form $(\omega^2, \omega\xi, \omega\eta, \omega\zeta, \eta\zeta, \zeta\xi, \xi\eta)$, the terms in ξ^2, η^2, ζ^2 being wanting.

Treating $(\xi, \eta, \zeta, \omega)$ as the coordinates of a point in space, the equation $\xi\eta\zeta + \omega^3 = 0$ is a cubic surface having a binode at each of the points $(\xi=0, \omega=0)$, $(\eta=0, \omega=0)$, $(\zeta=0, \omega=0)$, and the second equation is that of a quadric surface passing through these three points; hence the two equations together represent a sextic in space, or say a skew sextic, having a node at each of these three points. The equations $x : y : z = \mathrm{P} : \mathrm{Q} : \mathrm{R}$ establish a (1, 1) correspondence between the locus of O and this skew sextic. To find the degree of the locus we intersect it by the arbitrary line $ax + by + cz = 0$; viz., we intersect the skew sextic by the quadric surface $a\mathrm{P} + b\mathrm{Q} + c\mathrm{R} = 0$. This is a surface passing through the three nodes of the skew sextic, and it therefore besides intersects the skew sextic in $12 - 2.3, = 6$ points. Hence the locus is (as it should be) a sextic.

I consider the point $\eta=0$, $\zeta=0$, $\omega=0$, or say the point (1, 0, 0, 0), of the skew sextic. This is a node, and for the consecutive point on one branch we have $\eta : \zeta : \omega = m\epsilon : l\epsilon^2 : n\epsilon$, where ϵ is infinitesimal. The equation of the cubic surface gives $lm+n^3=0$, and the equation of the quadric surface gives $k_2k_3 . \frac{\omega}{\alpha} - k_1k_2\gamma\eta = 0$, that is, $k_3\omega = \alpha\gamma k_1\eta$, which, in fact, determines the ratio $l : m$; but it will be convenient to retain the equation in this form. For the corresponding values of (x, y, z) we have

$$\begin{aligned} x : y : z = \; & k_3\left(\alpha-\frac{1}{\alpha}\right)\frac{\omega}{\alpha} + k_1\left(\gamma-\frac{1}{\gamma}\right)\frac{\eta}{\alpha} \\ & : k_3\left(\alpha-\frac{1}{\alpha}\right)\beta\omega + k_1\left(\gamma-\frac{1}{\gamma}\right)\frac{\eta}{\beta} \\ & : k_3\left(\alpha-\frac{1}{\alpha}\right)\gamma\omega + k_1\left(\gamma-\frac{1}{\gamma}\right)\gamma\eta, \end{aligned}$$

which, writing for $k_3\omega$ its value $= \alpha\gamma k_1\eta$, become

$$\begin{aligned} x : y : z = \; & \left(\alpha-\frac{1}{\alpha}\right)\gamma + \left(\gamma-\frac{1}{\gamma}\right)\frac{1}{\alpha} = \alpha\gamma - \frac{\gamma}{\alpha} + \frac{\gamma}{\alpha} - \frac{1}{\gamma\alpha} \\ & : \left(\alpha-\frac{1}{\alpha}\right)\alpha\beta\gamma + \left(\gamma-\frac{1}{\gamma}\right)\frac{1}{\beta} : -\alpha + \frac{1}{\alpha} - \alpha\gamma^2 + \alpha \\ & : \left(\alpha-\frac{1}{\alpha}\right)\alpha\gamma^2 + \left(\gamma-\frac{1}{\gamma}\right)\gamma : \alpha^2\gamma^2 - \gamma^2 + \gamma^2 - 1 \end{aligned}$$

(the last set of values being obtained by aid of the relation $\alpha\beta\gamma = -1$); viz., we thus have

$$x : y : z = \frac{1}{\alpha\gamma}(-1+\alpha^2\gamma^2) : -\frac{1}{\alpha}(-1+\alpha^2\gamma^2) : (-1+\alpha^2\gamma^2),$$

that is,

$$x : y : z = \frac{1}{\gamma} : -1 : \alpha,$$

which are, in fact, the values belonging to one of the circular points at infinity. For the consecutive point on the other branch we should obtain in like manner $x : y : z = \gamma : -1 : \frac{1}{\alpha}$, which are the values belonging to the other circular point at infinity; viz., the node (1, 0, 0, 0) of the skew sextic corresponds to the circular points at infinity. But, in like manner, the other two nodes (0, 1, 0, 0) and (0, 0, 1, 0) each correspond to the circular points at infinity, or say we have in the skew sextic the three nodes each corresponding to one circular point at infinity, and the same three nodes each corresponding to the other circular point at infinity; viz., we thus prove that each of the circular points at infinity is a triple point on the locus of O.

[In order not to interrupt the demonstration I have assumed the

formulæ which, in the system of coordinates, x, y, z proportional to the perpendiculars on the sides of a triangle ABC, or where the equation of the line infinity is $x \sin A + y \sin B + z \sin C = 0$, give the circular points at infinity; viz., writing

$$\cos A + i \sin A,\ \cos B + i \sin B,\ \cos C + i \sin C = \alpha, \beta, \gamma,$$

the coordinates for the two points respectively are

$$x : y : z = -1 : \gamma : \frac{1}{\beta} \quad \text{and} \quad x : y : z = -1 : \frac{1}{\gamma} : \beta$$

$$= \frac{1}{\gamma} : -1 : \alpha \qquad\qquad = \gamma : -1 : \frac{1}{\alpha}$$

$$= \beta : \frac{1}{\alpha} : -1, \qquad\qquad = \frac{1}{\beta} : \alpha : -1,$$

the three values for each point being equivalent in virtue of the relation $\alpha\beta\gamma = -1$. This is, in fact, under a different form, the theorem given in my Smith's Prize paper for 1875; viz., the theorem was: If λ, μ, ν are the inclinations to a fixed line of the perpendiculars let fall from an interior point on the sides of the fundamental triangle ABC, then, in the system of trilinear coordinates in which the coordinates of a point P are proportional to the triangles PBC, PCA, PAB (or where the equation of the line infinity is $x+y+z=0$), the coordinates of the circular points at infinity are proportional, those of the one point to $e^{i\lambda} \sin(\mu-\nu)$, $e^{i\mu} \sin(\nu-\lambda)$, $e^{i\nu} \sin(\lambda-\mu)$, and those of the other point to $e^{-i\lambda} \sin(\mu-\nu)$, $e^{-i\mu} \sin(\nu-\lambda)$, $e^{-i\nu} \sin(\lambda-\mu)$.]

In the plane curve the lines drawn from A, B, C to the circular points at infinity are:

	To one point.	To other point.
From A,	$\alpha y + z = 0$,	$y + \alpha z = 0$;
„ B,	$\beta z + x = 0$,	$z + \beta x = 0$;
„ C,	$\gamma x + y = 0$,	$x + \gamma y = 0$.

Each of these lines, *quà* tangent at a triple point, meets the curve in the circular point at infinity counting four times, and in two other points. The corresponding points on the skew sextic should be a node counting twice, the two other nodes counting each once, and two other points. The proof that this is so would show that the points A, B, C are a triad of foci. There is also the question of the determination of the values of $(\xi, \eta, \zeta, \omega)$ which correspond to the nodes of the plane curve. But I have not further pursued the theory.

Addition.—Since writing the foregoing paper I have found that the relation between the nodes and foci (sum of angular distances of the foci = sum of angular distances of the nodes) may be expressed in a different form; viz., the triangle of the foci and the triangles of the nodes are circumscribed to a parabola (having its focus on the circle); and I have made in relation to the question the following further investigations:—

Considering a circle and a parabola having its focus at K, a point of the circle, then if, as usual, I, J are the circular points at infinity, we have IJK a triangle inscribed in the circle and circumscribed to the parabola; hence there exists a singly-infinite series of in- and circumscribed triangles, so that, drawing from a point A of the circle tangents to the parabola again meeting the circle in the points B and C respectively, BC will be a tangent to the parabola; or, what is the same thing, starting with the triangle ABC inscribed in the circle, we can, with the arbitrary point K on the circle as focus, describe a parabola touching the three sides of the triangle ABC; viz., the parabola described to touch two of the sides of the triangle will touch the third side.

Taking, then, a circle radius $\frac{1}{2}k$, and upon it the three points A, B, C determined by the angles 2α, 2β, 2γ respectively (viz., the coordinates of A are $x, y = \frac{1}{2}k\cos 2\alpha, \frac{1}{2}k\sin 2\alpha$, &c.), and a point K determined by the angle 2κ (suppose for a moment the origin is at K), the equation of a parabola having K for its focus will be

$$x^2+y^2 = (x\cos 2\theta + y\sin 2\theta - p)^2,$$

or, what is the same thing,

$$(x\sin 2\theta - y\cos 2\theta)^2 + 2p(x\cos 2\theta + y\sin 2\theta) - p^2 = 0,$$

where θ, p are in the first instance arbitrary; and the condition in order that $\xi x+\eta y+\zeta = 0$ may be a tangent is easily found to be

$$p(\xi^2+\eta^2) + 2\cos 2\theta\,.\,\xi + 2\sin 2\theta\,.\,\eta = 0.$$

It is to be shown that p, θ can be determined so that the parabola shall touch each of the lines BC, CA, AB.

Taking the origin at the centre, the equation of BC is

$$x\cos(\beta+\gamma) + y\sin(\beta+\gamma) - \tfrac{1}{2}k\cos(\beta-\gamma) = 0,$$

as is at once verified by showing that this equation is satisfied by the values $x, y = \frac{1}{2}k\cos 2\beta, \frac{1}{2}k\sin 2\beta$, and $= \frac{1}{2}k\cos 2\gamma, \frac{1}{2}k\sin 2\gamma$.

Hence, transforming to the point K as origin, the equation is

$$[x+\tfrac{1}{2}k\cos 2\kappa]\cos(\beta+\gamma) + [y+\tfrac{1}{2}k\sin 2\kappa]\sin(\beta+\gamma)$$

viz., this is $\qquad -\tfrac{1}{2}k\cos(\beta-\gamma) = 0;$

$$x\cos(\beta+\gamma) + y\sin(\beta+\gamma) - \tfrac{1}{2}k[\cos(\beta-\gamma) - \cos(\beta+\gamma-2\kappa)] = 0;$$

or, finally, it is

$$x\cos(\beta+\gamma) + y\sin(\beta+\gamma) - k\sin(\kappa-\beta)\sin(\kappa-\gamma) = 0.$$

Hence the condition of contact with the line BC is

$$p = 2k\sin(\kappa-\beta)\sin(\kappa-\gamma)\cos(2\theta-\beta-\gamma);$$

and, similarly, the condition of contact with the line CA is

$$p = 2k\sin(\kappa-\gamma)\sin(\kappa-\alpha)\cos(2\theta-\gamma-\alpha);$$

viz., these conditions determine the unknown quantities p, θ. It is at once seen that we have

$$2\theta-\beta-\gamma = \tfrac{1}{2}\pi-(\kappa-\alpha), \quad \text{that is,} \quad 2\theta = \tfrac{1}{2}\pi-\kappa+\alpha+\beta+\gamma;$$

and then $$p = 2k \sin(\kappa-\alpha) \sin(\kappa-\beta) \sin(\kappa-\gamma);$$

and from symmetry we see that the parabola touches also the side AB.

Suppose, next, F, G are points on the circle determined by the angles $2f$, $2g$; retaining p and θ to denote their values,

$p = 2k \sin(\kappa-\alpha) \sin(\kappa-\beta) \sin(\kappa-\gamma)$, and $2\theta = \tfrac{1}{2}\pi-\kappa+\alpha+\beta+\gamma$,

the condition in order that FG may be a tangent is

$$p = 2k \sin(\kappa-f) \sin(\kappa-g) \cos(2\theta-f-g);$$

viz., determining h by the equation

$$\alpha+\beta+\gamma = f+g+h,$$

this is $$p = 2k \sin(\kappa-f) \sin(\kappa-g) \sin(\kappa-h),$$

or, what is the same thing,

$$\sin(\kappa-\alpha) \sin(\kappa-\beta) \sin(\kappa-\gamma) = \sin(\kappa-f) \sin(\kappa-g) \sin(\kappa-h);$$

viz., this equation, considering therein h as standing for $\alpha+\beta+\gamma-f-g$, is the relation which must subsist between f, g in order that the line FG may be a tangent to the parabola. And then, h being determined as above, and the point H on the circle being determined by the angle $2h$, it is clear that the lines GH, HF will also be tangents to the parabola; viz., FGH will be an in- and circumscribed triangle, provided only f, g, h satisfy the above-mentioned two equations. The last of these, if f, g, h satisfy only the relation $\alpha+\beta+\gamma = f+g+h$, serves to determine κ; and then, θ and κ denoting as above, the equation of the parabola is $$x^2+y^2 = (x \cos 2\theta + y \sin 2\theta - p)^2;$$

and it thus appears that the condition in question, $\alpha+\beta+\gamma = f+g+h$, is equivalent to the condition that the triangles ABC, FGH shall be circumscribed to the same parabola.

It is to be remarked that the distances KA, KB, &c. are equal to $k \sin(\kappa-\alpha)$, $k \sin(\kappa-\beta)$, &c.; hence the condition

$$\sin(\kappa-\alpha) \sin(\kappa-\beta) \sin(\kappa-\gamma) = \sin(\kappa-f) \sin(\kappa-g) \sin(\kappa-h)$$

becomes $$\mathrm{KA} \,.\, \mathrm{KB} \,.\, \mathrm{KC} = \mathrm{KF} \,.\, \mathrm{KG} \,.\, \mathrm{KH};$$

viz, the focus K is a point on the circle such that the product of its (linear) distances from the foci A, B, C is equal to the product of its (linear) distances from the nodes F, G, H.

It is to be remarked that the foregoing equation in κ determines a single position of the point K; viz., it determines $\tan\kappa$, and therefore $\sin 2\kappa$ and $\cos 2\kappa$, linearly. The equation is, in fact, a cubic equation in $\tan\kappa$, satisfied identically by $\tan\kappa = i$ and $\tan\kappa = -i$, and therefore reducible to a linear equation.

Write for a moment $\tan\kappa = \omega$, and

$$(\tan\kappa - \tan\alpha)(\tan\kappa - \tan\beta)(\tan\kappa - \tan\gamma) = \omega^3 - p\omega^2 + q\omega - r,$$

$$(\tan\kappa - \tan f)(\tan\kappa - \tan g)(\tan\kappa - \tan h) = \omega^3 - p'\omega^2 + q'\omega - r';$$

also $$\mathrm{M} = \cos f \cos g \cos h \div \cos\alpha \cos\beta \cos\gamma.$$

Then we have $\omega^3 - p\omega^2 + q\omega - r = \mathrm{M}(\omega^3 - p'\omega^2 + q'\omega - r)$,

where $p = r + \mathrm{M}(r' - p)$, $q = 1 + \mathrm{M}(q' - 1)$.

Substituting these values, the equation becomes

$$\omega^3 - r\omega^2 + \omega - r = \mathrm{M}(\omega^3 - r'\omega^2 + \omega - r'),$$

viz., dividing by $\omega^2 + 1$, this is $\omega - r = \mathrm{M}(\omega - r')$; or substituting for r, r', M their values,

$$(\cos\alpha\cos\beta\cos\gamma - \cos f \cos g \cos h)\tan\kappa$$
$$= (\sin\alpha\sin\beta\sin\gamma - \sin f \sin g \sin h),$$

which is the value of $\tan\kappa$, and then

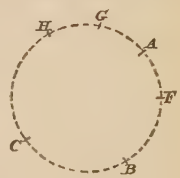
$$\sin 2\kappa = \frac{2\sin\kappa}{1+\tan^2\kappa}, \quad \cos 2\kappa = \frac{1-\tan^2\kappa}{1+\tan^2\kappa}.$$

It may be further noticed that if the parabola intersect the circle in a point L, and the tangent at L to the parabola again meet the circle in M, then, if $2l$, $2m$ are the angles for the points L, M, we have l, m, m for values of f, g, h, whence l, m are determined by the equations

$$l + 2m = \alpha + \beta + \gamma,$$
$$\sin(\kappa - l)\sin^2(\kappa - m) = \sin(\kappa - \alpha)\sin(\kappa - \beta)\sin(\kappa - \gamma);$$

but as the circle intersects the parabola not only in two real points, but in two other imaginary points, there is no simple formula for the determination of l and m.

To determine the linkage when the nodes are given, suppose in the generation by O, the vertex of the triangle OB_1C_1, we have O at the node F, then, if τ, σ are the distances of C, B from the node in question, we have, as in the memoir,



$$(b_1^2 + \tau^2 - a_2^2)\, c_1\sigma = (c_1^2 + \sigma^2 - a_3^2)\, b_1\tau,$$

that is,

$$(b_1^2 - a_2^2)\, c_1\sigma - (c_1^2 - a_3^2)\, b_1\tau + \sigma\tau(c_1\tau - b_1\sigma) = 0,$$

or, what is the same thing,

$$(b_1^2 - a_2^2)\, c\sigma - (c_1^2 - a_3^2)\, b\tau + \sigma\tau(c\tau - b\sigma) = 0.$$

Suppose, as in the figure, that F is between B and A; then, if $AF = \rho$, we have $c\tau = b\sigma + a\rho$, and the equation becomes

$$(b_1^2 - a_2^2)\, c\sigma - (c_1^2 - a_3^2)\, b\tau + a\rho\sigma\tau = 0;$$

and similarly, if, as in the figure, G is on the other side of A, that is,

between A and C, and if ρ', σ', τ' be the distances AG, BG, CG, then $b\sigma' = c\tau' + a\rho'$, that is, $c\tau' - b\sigma' = -a\rho'$, and the corresponding equation is $$(b_1^2 - a_2^2)\, c\sigma' - (c_1^2 - a_3^2)\, b\tau' - a\rho'\sigma'\tau' = 0.$$

We hence find

$$(b_1^2 - a_2^2)\, c\,(\sigma\tau' - \sigma'\tau) + a\tau\tau'(\rho\sigma + \rho'\sigma') = 0,$$
$$(c_1^2 - a_3^2)\, b\,(\sigma\tau' - \sigma'\tau) + a\sigma\sigma'(\rho\tau + \rho'\tau') = 0.$$

But we have $\mathrm{BG}\,.\,\mathrm{CF} = \mathrm{BF}\,.\,\mathrm{CG} + \mathrm{BC}\,.\,\mathrm{FG}$,

that is, $\sigma'\tau = \sigma\tau' + a\,.\,\mathrm{FG}$, or $\sigma\tau' - \sigma'\tau = -a\,.\,\mathrm{FG}$;

also, $\rho\sigma + \rho'\sigma' = \mathrm{AF}\,.\,\mathrm{BF} + \mathrm{AG}\,.\,\mathrm{BG}, = \mathrm{FG}\,.\,\mathrm{CH}, = \mathrm{FG}\,.\,\tau''$,

and $\rho\tau + \rho'\tau' = \mathrm{AF}\,.\,\mathrm{CF} + \mathrm{AG}\,.\,\mathrm{CG}, = \mathrm{FG}\,.\,\mathrm{BH}, = \mathrm{FG}\,.\,\sigma''$,

as may be shown without difficulty (ρ'', σ'', τ'' being the distances AH, BH, CH). Hence the equations become

$$c\,(b_1^2 - a_2^2) - \tau\tau'\tau'' = 0,$$
$$b\,(c_1^2 - a_3^2) - \sigma\sigma'\sigma'' = 0,$$

showing that, the foci being as in the figure, $b_1^2 - a_2^2$ and $c_1^2 - a_3^2$ are each of them positive; viz., that in the generation by the triangle $\mathrm{OC_1B_1}$, the radial bars a_2, a_3 are shorter than the sides b_1, c_1 respectively. Substituting for b_1, &c. the values $k_1 \sin \mathrm{B}$, &c.; also, instead of k_1, k_2, k_3, introducing the quantities λ_1, λ_2, λ_3, where

$$k_1,\ k_2,\ k_3 = \lambda_1 \sin \mathrm{A},\ \lambda_2 \sin \mathrm{B},\ \lambda_3 \sin \mathrm{C},$$

these equations become

$$\sin^2 \mathrm{A} \sin^2 \mathrm{B}\,.\,c\,(\lambda_1^2 - \lambda_2^2) = \tau\tau'\tau'',$$
$$\sin^2 \mathrm{A} \sin^2 \mathrm{C}\,.\,b\,(\lambda_1^2 - \lambda_3^2) = \sigma\sigma'\sigma'';$$

or, as these may be written, putting for shortness $\mathrm{M} = \sin \mathrm{A} \sin \mathrm{B} \sin \mathrm{C}$,

$$\mathrm{M}^2 k^2 (\lambda_1^2 - \lambda_2^2) = c\,.\,\tau\tau'\tau'',$$
$$\mathrm{M}^2 k^2 (\lambda_1^2 - \lambda_3^2) = b\,.\,\sigma\sigma'\sigma''.$$

All the quantities have so far been regarded as positive, and the formulæ are applicable to the particular figure; but, to present them in a form applicable to any order of the nodes and foci, we have only to write the equations in the forms

$$\mathrm{M}^2 (\lambda_1^2 - \lambda_2^2) = k^2 \sin(\alpha - \beta) \sin(f - \gamma) \sin(g - \gamma) \sin(h - \gamma),$$
$$\mathrm{M}^2 (\lambda_1^2 - \lambda_3^2) = k^2 \sin(\alpha - \gamma) \sin(f - \beta) \sin(g - \beta) \sin(h - \beta);$$

and these may be replaced by the system

$$\mathrm{M}^2 (\lambda_2^2 - \lambda_3^2) = k^2 \sin(\beta - \gamma) \sin(f - \alpha) \sin(g - \alpha) \sin(h - \alpha),$$
$$\mathrm{M}^2 (\lambda_3^2 - \lambda_1^2) = k^2 \sin(\gamma - \alpha) \sin(f - \beta) \sin(g - \beta) \sin(h - \beta),$$
$$\mathrm{M}^2 (\lambda_1^2 - \lambda_2^2) = k^2 \sin(\alpha - \beta) \sin(f - \gamma) \sin(g - \gamma) \sin(h - \gamma),$$

since the first of these equations is implied in the other two; and then, reverting to the original form, we may write

$$M^2k^2(\lambda_2^2-\lambda_3^2) = BC\,.\,FA\,.\,GA\,.\,HA,$$
$$M^2k^2(\lambda_3^2-\lambda_1^2) = CA\,.\,FB\,.\,GB\,.\,HB,$$
$$M^2k^2(\lambda_1^2-\lambda_2^2) = AB\,.\,FC\,.\,GC\,.\,HC,$$

it being understood that the distances BC, FA, &c., which enter into these equations, are not all positive, but that they stand for $k\sin(\beta-\gamma)$, $k\sin(f-a)$, &c., and that their signs are to be taken accordingly. Or, again, these may be written

$$BC\,(c_2^2-b_3^2) = FA\,.\,GA\,.\,HA,$$
$$CA\,(a_3^2-c_1^2) = FB\,.\,GB\,.\,HB,$$
$$AB\,(b_1^2-a_2^2) = FC\,.\,GC\,.\,HC,$$

where the signs are as just mentioned. We may say that $\pm(c_2^2-b_3^2)$ is the modulus for the focus A, and the formula then shows that this modulus, taken positively, is equal to the product of the distances FA, GA, HA of A from the three nodes respectively, divided by BC, the distance of the other two foci from each other.

On the Bicursal Sextic. *By* Prof. Cayley.

[*Read March 10th*, 1876.]

In the paper "On the Mechanical Description of Certain Sextic Curves," "Proceedings London Mathematical Society," vol. iv. (1872), pp. 105—111, I obtained the bicursal sextic as a rational transformation of a binodal quartic. The theory was in effect as follows: taking Ω, P, Q, R, each of them a function of λ, μ of the form $(*\nabla 1, \lambda)^2(1, \mu)^2$, and considering (λ, μ) as connected by the equation $\Omega=0$ (viz., λ, μ being coordinates, this represents a binodal quartic), then, if we assume $x : y : z = P : Q : R$, the locus of the point (x, y, z) is a curve rationally connected with the binodal quartic (viz., the points of the two curves have with each other a $(1, 1)$ correspondence); whence the locus in question, say the curve $U = 0$, is bicursal. The degree is obtained as the number of the intersections of the curve by an arbitrary line, or, what is the same thing, the number of the variable intersections of the corresponding $\lambda\mu$-curves

$$\Omega = 0, \quad \alpha P+\beta Q+\gamma R = 0,$$

viz., each of these being a quartic curve having the same two nodes, the nodes each count as 4 intersections, and the number of the remaining intersections is $4.4-2.4$, $=8$, and thus the curve $U=0$ is in general of the order 8. But if the curves $\Omega=0$, $P=0$, $Q=0$, $R=0$ have (besides the nodes) k common intersections, then these are also fixed intersections of the two curves $\Omega=0$, $\alpha P+\beta Q+\gamma R=0$, and the number of variable intersections is reduced to $8-k$; we have thus $8-k$ as the order of the curve $U=0$, and, in particular, if $k=2$, then the curve is a bicursal sextic.

The theory assumes a different and more simple form if, in the several functions Ω, P, Q, R, we suppose that the terms in λ^2, μ^2 are wanting. The curves $\Omega=0$, $P=0$, $Q=0$, $R=0$ are here cubics having two common points; the curve $U=0$, *quâ* rational transformation of the cubic $\Omega=0$, is still a bicursal curve; but its order is given as the number of the variable intersections of the cubics

$$\Omega=0, \quad \alpha P+\beta Q+\gamma R=0,$$

viz., this is $=3.3-2$, $=7$. But if the curves $\Omega=0$, $P=0$, $Q=0$, $R=0$ have (besides the before-mentioned two common points) k other common points, then the number of the variable intersections is $=7-k$: and this is therefore the order of the curve $U=0$, and in particular, if $k=1$, then the curve is a bicursal sextic. And, in the present paper, I consider the binodal sextic as thus obtained, viz., as given by the equations $\Omega=0$, $x:y:z=P:Q:R$, where $\Omega=0$, $P=0$, $Q=0$, $R=0$ are cubics, having (in all) three common points.

The bicursal sextic has in general 9 nodes; but 3 of these may unite together into a triple point: this will be the case if, in the series of curves $\alpha P+\beta Q+\gamma R=0$, there are any two curves which have 3 common intersections with the curve $\Omega=0$. (Observe that we throughout disregard the 3 common points of the curves $\Omega=0$, $P=0$, $Q=0$, $R=0$, and attend only to the 6 variable points of intersection of the curves $\Omega=0$ and $\alpha P+\beta Q+\gamma R=0$,—the meaning is, that there are two curves of the series such that, attending only to the 6 variable intersections of each of them with the curve $\Omega=0$, there are three common intersections.) For, supposing the two curves to be $\alpha P+\beta Q+\gamma R=0$ and $\alpha' P+\beta' Q+\gamma' R=0$, then any curve whatever $\alpha P+\beta Q+\gamma R+\theta(\alpha' P+\beta' Q+\gamma' R)=0$ has the same three intersections with the curve $\Omega=0$, say these are the points A_1, A_2, A_3, the coordinates of which are independent of θ. Hence the line

$$(\alpha x+\beta y+\gamma z)+\theta(\alpha' x+\beta' y+\gamma' z)=0$$

intersects the curve $U=0$ in six points, three of which, as corresponding to the points A_1, A_2, A_3, are independent of θ, viz., they are the same three points for any line whatever of the series; and this means

that the curve $U = 0$ has at the point

$$(\alpha x+\beta y+\gamma z = 0, \quad \alpha' x+\beta' y+\gamma' z = 0)$$

a triple point; and that to this correspond the three points A_1, A_2, A_3.

We may in the series of lines $\alpha x+\beta y+\gamma z+\theta(\alpha' x+\beta' y+\gamma' z) = 0$ rationally determine θ so that one of the three variable points of intersection shall correspond to A_1, A_2, or A_3; viz., θ must be such that the curve $\alpha P+\beta Q+\gamma R+\theta(\alpha' P+\beta' Q+\gamma' R) = 0$ shall touch the curve $\Omega = 0$ at one of the points A_1, A_2, A_3. The three lines thus determined are the three tangents to the curve at the triple point: and the three branches may be considered as corresponding to the three points A_1, A_2, A_3, respectively.

There is no loss of generality in assuming that the triple point is the point $(x = 0, z = 0)$; the condition then simply is that the curves $P = 0$, $R = 0$ shall have three common intersections with the curve $\Omega = 0$; and the tangents at the triple point are $x+\theta z = 0$, θ being so determined that one of the three variable points of intersection shall correspond to one of the three points A_1, A_2, A_3: in particular, if this is the case for the line $x = 0$, then this line will be one of the tangents at the triple point.

The bicursal sextic may have a second triple point, viz., three other nodes may unite together into a triple point. The theory is precisely the same: we must have two other curves $\alpha P+\beta Q+\gamma R = 0$, $\alpha' P+\beta' Q+\gamma' R = 0$, having with the curve $\Omega = 0$ three common intersections B_1, B_2, B_3: there is then a second triple point

$$(\alpha x+\beta y+\gamma z = 0, \quad \alpha' x+\beta' y+\gamma' z = 0);$$

and, to find the tangents at this point, we must determine θ so that one of the variable points of intersection of the line

$$\alpha x+\beta y+\gamma z+\theta(\alpha' x+\beta' y+\gamma' z) = 0$$

with the sextic shall correspond with B_1, B_2, or B_3; viz., θ must be such that the curve $\alpha P+\beta Q+\gamma R+\theta(\alpha' P+\beta' Q+\gamma' R) = 0$ shall touch the curve $\Omega = 0$ at one of the points B_1, B_2, B_3. In particular, if, as before, the curves $P = 0$, $R = 0$ have three common intersections with the curve $\Omega = 0$, and if, moreover, the curves $Q = 0$, $R = 0$ have three common intersections with the curve $\Omega = 0$, then the bicursal sextic will have the two triple points $(x = 0, z = 0)$ and $(y = 0, z = 0)$; and it may further happen that the line $x = 0$ is a tangent at the first triple point, and the line $y = 0$ a tangent at the second triple point. The sextic may in like manner have a third triple point, but this is a special case which I do not at present consider.

I write for greater convenience $\frac{\lambda}{\nu}$, $\frac{\mu}{\nu}$ in place of λ, μ, so as to make Ω, P, Q, R each of them a homogeneous cubic function of (λ, μ, ν); and

I give to these functions, not the most general values belonging to a bicursal sextic with two triple points, but the values in the form obtained for them, as appearing further on, in the problem of three-bar motion; viz., the equations $\Omega = 0$, $x : y : z = P : Q : R$ are taken to be

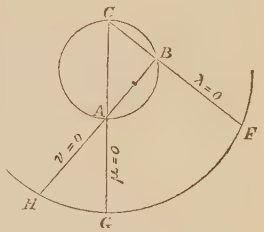
$$\nu\,(h\nu\lambda + f\lambda^2) + \mu\,(g\nu^2 + e\nu\lambda + g\lambda^2) + \mu^2\,(f\nu + h\lambda) = 0,$$

$$x : y : z = \lambda\mu\,(a\lambda + b\mu) : \nu^2\,(c\lambda + d\mu) : \lambda\mu\nu.$$

The four curves $\Omega = 0$, $P = 0$, $Q = 0$, $R = 0$ have thus the three common intersections

$(\mu = 0, \nu = 0)$ $(\nu = 0, \lambda = 0)$ $(\lambda = 0, \mu = 0)$,

represented in the figure by the points A, B, C; the curve drawn in the figure is the curve $\Omega = 0$, and the points F, G, H are the third points of intersection of the cubic with the lines BC, CA, AB respectively.



The equation $P + \theta R = 0$ is here $\lambda\mu\,(a\lambda + b\mu + \theta\nu) = 0$, which intersects $\Omega = 0$ in the points C, A, G, C. B, F, and the three intersections by the line $a\lambda + b\mu + \theta\nu = 0$: viz., excluding the fixed points A, B, C, the six intersections are C, F, G, and the three intersections by the line. Hence, of the six intersections, we have C, F, G independent of θ, or we have $(x = 0,\ z = 0)$ a triple point, say I, corresponding to the three points C, F, G, viz., these are the points

$$(\lambda, \mu, \nu) = (0, 0, 1),\ (0, g, -f),\ (h, 0, -f). \qquad (C, F, G).$$

The equation $Q + \theta R = 0$ is $\nu\,(c\lambda\nu + d\mu\nu + \theta\lambda\mu) = 0$; viz., the line $\nu = 0$ meets $\Omega = 0$ in the three points A, B, H, and the conic $c\lambda\nu + d\mu\nu + \theta\lambda\mu = 0$ meets $\Omega = 0$ in the points A, B, C, and three other points: hence, rejecting the points A, B, C, the six points of intersection are the points A, B, H, and the three variable points of intersection by the conic; or we have $(y = 0,\ z = 0)$ a triple point, say J, corresponding to the three points A, B, H, viz., these are the points

$$(\lambda, \mu, \nu) = (1, 0, 0),\ (0, 1, 0),\ (h, -g, 0), \qquad (A, B, H).$$

To find the tangents at the triple point I, these are $x + \theta z = 0$ where θ is to be successively determined by the conditions that the line $a\lambda + b\mu + \theta\nu = 0$ shall pass through the points C, F, G,* viz., we thus

* Observe the somewhat altered form of the condition: θ is to be determined so that the cubic $\lambda\mu\,(a\lambda + b\mu + \theta\nu) = 0$ shall touch the cubic $\Omega = 0$ at one of the points C, F, G: but, as the first-mentioned cubic breaks up, and the component curve $a\lambda + b\mu + \theta\nu = 0$ does not pass through any one of these points, this can only mean that θ shall be so determined as that the line shall pass through one of these points, viz., that there shall be at the point, not a proper contact, but a double intersection, arising from a node of the cubic $\lambda\mu\,(a\lambda + b\mu + \theta\nu) = 0$. And the like case happens for the other triple point; viz., there the cubic $\nu\,(c\nu\lambda + d\nu\mu + \theta\lambda\mu) = 0$ is to touch the cubic $\Omega = 0$ at one of the points A, B, H; the component conic $c\nu\lambda + d\nu\mu + \theta\lambda\mu = 0$ passes through the points A and B but not through H; hence the conditions for θ are that the conic shall touch the cubic at A or B, or that it shall pass through H.

have

$\theta = 0$, $x = 0$ tangent corresponding to point C (0, 0, 1),

$\theta = +\frac{bg}{f}$, $fx + bgz = 0$ „ „ „ F (0, g, $-f$),

$\theta = +\frac{ah}{f}$, $fx + ahz = 0$ „ „ „ G (h, 0, $-f$).

And similarly at the triple point J the tangents are $y + \theta z = 0$, where θ is to be successively determined by the conditions that the conic $c\nu\lambda + d\nu\mu + \theta\lambda\mu = 0$ shall pass through the point H, and shall touch the cubic at the points A, B; viz., we thus have

$\theta = 0$, $y = 0$ tangent corresponding to point H (h, $-g$, 0),

$\theta = \frac{cg}{f}$, $fy + cgz = 0$ „ „ „ A (1, 0, 0),

$\theta = \frac{dh}{f}$, $fy + dhz = 0$ „ „ „ B (0, 1, 0).

The two last values of θ are obtained by the consideration that the equations of the tangents to $\Omega = 0$ at the points A, B respectively, are $g\mu + f\nu = 0$, $h\lambda + f\nu = 0$, where λ, μ, ν are current coordinates of a point on the tangent: it may be added that the equation of the tangent at the point C is $h\lambda + g\mu = 0$.

The three-bar curve may be represented by means of a system of equations of the last-mentioned form $(x : y : z = \lambda\mu\,(a\lambda + b\mu) : \nu^2\,(c\lambda + d\mu) : \lambda\mu\nu$, where λ, μ, ν are connected as above); viz., taking X, Y as ordinary rectangular coordinates, x, y, and z are here the circular coordinates $\frac{x}{z} = \mathrm{X} + i\mathrm{Y}$, $\frac{y}{z} = \mathrm{X} - i\mathrm{Y}$, and $z = 1$; and the parameters $\frac{\lambda}{\nu}$, $\frac{\mu}{\nu}$ denote like functions $\cos\theta + i\sin\theta$, $\cos\phi + i\sin\phi$ of angles which are the inclinations of two bars to a fixed line. Using, for convenience, Figure 2 of my paper on Three-bar Motion, the curve is considered as the locus of the vertex O of the triangle $\mathrm{OC_1B_1}$, connected by the bars $\mathrm{C_1C}$ and $\mathrm{B_1B}$ with the fixed points B and C respectively; and we have $\mathrm{CC_1} = a_2$, $\mathrm{OC_1} = b_1$, $\mathrm{OB_1} = c_1$, $\mathrm{C_1B_1} = a_1$, $\mathrm{B_1B} = a_3$. BC (to avoid confusion with the foregoing notation of the present paper, instead of calling it a) I take $= a_0$: the angle $\mathrm{OC_1B_1}$ is $= \mathrm{C_1}$, and $\cos\mathrm{C_1} + i\sin\mathrm{C_1}$ is taken $= \gamma$.

Hence, taking the origin at C, the axis of X coinciding with CB, and that of Y at right angles to it, taking also θ, ϕ, ψ for the inclinations of $\mathrm{CC_1}$, $\mathrm{C_1B_1}$, and $\mathrm{B_1B}$ to CB, we have

$$a_2\cos\theta + a_1\cos\phi - a = -a_3\cos\psi,$$
$$a_2\sin\theta + a_1\sin\phi = a_3\sin\psi;$$

viz., writing $\cos\theta+i\sin\theta=\lambda$, $\cos\phi+i\sin\phi=\mu$, these give

$$a_2\lambda+a_1\mu-a_0=-a_3(\cos\psi-i\sin\psi),$$
$$a_2\frac{1}{\lambda}+a_1\frac{1}{\mu}-a_0=-a_3(\cos\psi+i\sin\psi),$$

that is,
$$(a_2\lambda+a_1\mu-a)\left(a_2\frac{1}{\lambda}+a_1\frac{1}{\mu}-a_0\right)-a_3^2=0;$$

viz.,

$$(a_0^2+a_1^2+a_2^2-a_3^2)+a_1a_2\left(\frac{\mu}{\lambda}+\frac{\lambda}{\mu}\right)-a_0a_2\left(\lambda+\frac{1}{\lambda}\right)-a_0a_1\left(\mu+\frac{1}{\mu}\right)=0,$$

for the relation between the parameters λ, μ. And then

$$X=a_2\cos\theta+b_1\cos(\phi+C_1),$$
$$Y=a_2\sin\theta+b_1\sin(\phi+C_1);$$

viz., if x, $y=X+iY$, $X-iY$, then

$$\begin{cases} x=a_2\lambda+b_2\gamma_1\mu, \\ y=a_2\frac{1}{\lambda}+b_2\frac{1}{\gamma_1\mu}, \end{cases}$$

which equations determine the coordinates (x, y) in terms of the parameters λ, μ connected by the foregoing relation.

Writing for homogeneity $\frac{\lambda}{\nu}$, $\frac{\mu}{\nu}$ in place of λ, μ, and $\frac{x}{z}$, $\frac{y}{z}$ in place of x, y, the equations become

$$(a_0^2+a_1^2+a_2^2-a_3^2)\lambda\mu\nu$$
$$+a_1a_2(\lambda^2+\mu^2)\nu-a_0a_2\mu(\nu^2+\lambda^2)-a_0a_1\lambda(\mu^2+\nu^2)=0,$$

and
$$x:y:z=(a_2\lambda+b_2\gamma_1\mu)\lambda\mu:\left(\frac{b_2}{\gamma_1}\lambda+a_2\mu\right)\nu^2:\lambda\mu\nu.$$

Comparing with the foregoing equations

$$e\lambda\mu\nu+f(\lambda^2+\mu^2)\nu+g(\nu^2+\lambda^2)\mu+h(\mu^2+\nu^2)\lambda=0,$$

and
$$x:y:z=(a\lambda+b\mu)\lambda\mu:(c\lambda+d\mu)\nu^2:\lambda\mu\nu,$$

the equations agree together, and we have

$$\begin{cases} e=a_0^2+a_1^2+a_2^2-a_3^2, \\ f=+a_1a_2, \\ g=-a_0a_2, \\ h=-a_0a_1, \\ a=a_2, \\ b=b_1\gamma_1, \\ c=\frac{b_1}{\gamma_1}, \\ d=a_2. \end{cases}$$

The tangents at the triple points thus are

$$x = 0, \qquad y = 0,$$
$$a_1 x - a_0 b_1 \gamma_1 z = 0, \qquad a_1 y - \frac{a_0 b_1}{\gamma_1} z = 0,$$
$$x - a_0 z = 0, \qquad y - a_0 z = 0;$$

viz., restoring the rectangular coordinates, and for γ substituting the value $\cos C + i \sin C$, for a_0 writing a, and taking $b = \frac{ab_1}{a_1}$

$$X + iY = 0, \qquad X - iY = 0,$$
$$X + iY = b(\cos C + i \sin C), \qquad X - iY = b(\cos C - i \sin C),$$
$$X + iY = a_0, \qquad X - iY = a_0;$$

viz., the first two intersect in the point (0, 0), the second two in the point ($b \cos C$, $b \sin C$), the third two in the point (a, 0): the first and third of these are the points B and C, the second of them is the point A of the figure; viz., the formulæ give the point A, forming, with B and C, a triad of foci.

April 13*th*, 1876.

Prof. H. J. S. SMITH, F.R.S., President, in the Chair.

The Rev. Dr. Logan was proposed for election.

Prof. Henrici having taken the Chair, the President gave an account of M. Charles Hermite's "Note sur un Théorème d'Eisenstein." He then spoke on "The Aspects of Circles on a Plane or on a Sphere," pointing out the relation between his results and those obtained by Prof. Cayley in his researches on "Trees." The President then made a few remarks on a question in Crystallography. Mr. Tucker read a portion of an abstract of a paper by Prof. Sturm "On Correlation in Space," which had been drawn up by Dr. Hirst.

The following presents were received:—

Carte de Visite likenesses of Drs. Kronecker and Klein.

"Monatsbericht," Dec. 1875, Jan. 1876.

"Reprint of the Mathematics from the Educational Times," Vol. xxiv.

"Crelle," Band 81, drittes Heft.

"Bulletin des Sciences Mathématiques et Astronomiques," Tome x$^{\text{ieme}}$, Janvier 1876.

"Bücher-verzeichniss von Mayer und Müller in Berlin," Katalog xv.

"Proceedings of the Royal Society," Vol. xxiv., No. 167.

"Nieuw Archief voor Wiskunde," deel 1, stuk i. ii., Amsterdam, 1875.

"Journal of Education," April 1876.

Sur un Théorème d'Eisenstein. *Par* M. Hermite.

[*Read April 13th*, 1876.]

M. Heine en donnant la démonstration du théorème célèbre d'Eisenstein, sur les développements en série des racines des équations algébriques, $f(y, x) = 0$, dans le Journal de Crelle (t. 48, page 267), y a ajouté cette remarque extrêmement importante, qu'on peut ramener les coefficients supposés commensurables d'un tel développement, à être tous entiers, sauf le premier, par le changement de x en kx.* C'est une simplification de la méthode employée par l'éminent géomètre, que je me propose d'indiquer en peu de mots. Considérons d'abord l'ensemble des divers développements ordonnés suivant les puissances entières et positives de la variable, qu'on peut tirer de l'équation proposée. J'observerai avec M. Heine, que si deux ou plusieurs d'entre eux, commençant par les mêmes termes, ont la partie commune

$$a+bx+cx^2+\ldots+kx^p,$$

la transformée: $\mathrm{F}(z, x) = 0$, obtenue en posant:

$$y = a+bx+cx^2+\ldots+kx^p+zx^{p+1},$$

aura cette propriété, que, pour $x=0$, toutes les racines seront nécessairement inégales. Cela étant, et désignant l'une d'elles supposée commensurable par z_0, je raisonnerai sur l'équation: $\mathrm{F} = (z+z_0, x) = 0$, qui sera par conséquent de la forme suivante:

$$\begin{aligned} &m_1x+m_2x^2+m_3x^3+\ldots \\ &\quad +z\ (n+n_1x+n_2x^2+\ldots) \\ &\quad +z^2\ (p+p_1x+p_2x^2+\ldots) \\ &\quad \ldots\quad \ldots\quad \ldots\quad \ldots\quad \ldots \\ &\quad +z^\mu(s+s_1x+s_2x^2+\ldots) = 0, \end{aligned}$$

les coefficients étant des nombres entiers et n devant essentiellement être supposé différent de zéro. Soit maintenant: $z = nu$ et $x = n^2t$; il viendra après avoir divisé par n^2:

$$\begin{aligned} &m_1t+n^2m_2t^2+\ldots \\ &\quad +u\ (1+nn_1t+n^3n_2t^2+\ldots) \\ &\quad +u^2\ (p+n^2p_1t+n^4p_2t^2+\ldots) \\ &\quad \ldots\quad \ldots\quad \ldots\quad \ldots\quad \ldots \\ &\quad +u^\mu n^{\mu-2}\ (s+n^2s_1t+n^4s_2t^2+\ldots) = 0, \end{aligned}$$

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* Note added by the permission of M. Hermite.—This remark had already been made by Eisenstein himself. His words are, "Endlich kann statt x immer ein solches Vielfache von x gesetzt werden, dass alle Coefficienten der Reihe in ganze Zahlen uebergehen." (See Eisenstein's note in the "Monatsberichte" of the Berlin Academy for July, 1852, p. 441; or the extract from it in an earlier paper of M. Heine's in "Crelle's Journal," vol. xlv., p. 285.)—H. J. S. Smith.

relation que j'écrirai ainsi :

$$u = -\frac{m_1 t + n^2 m_2 t^2 + \dots}{1 + n n_1 t + \dots}$$
$$- u^2 \frac{p + n^2 p_1 t + \dots}{1 + n n_1 t + \dots}$$
$$\dots \quad \dots \quad \dots$$
$$- u^\mu n^{\mu-2} \frac{s + n^2 s_1 t + \dots}{1 + n n_1 t + \dots},$$

ou encore :

$$u = \mathrm{M}_1 t + \mathrm{M}_2 t^2 + \dots$$
$$+ u^2 (\mathrm{P} + \mathrm{P}_1 t + \mathrm{P}_2 t^2 + \dots)$$
$$+ u^3 (\mathrm{Q} + \mathrm{Q}_1 t + \mathrm{Q}_2 t^2 + \dots)$$
$$\dots \quad \dots \quad \dots \quad \dots \quad \dots$$
$$+ u^\mu (\mathrm{S} + \mathrm{S}_1 t + \mathrm{S}_2 t^2 + \dots),$$

en observant que les séries infinies introduites dans le second membre, ont toutes pour coefficients des nombres entiers. Faisant donc :

$$u = a_1 t + a_2 t^2 + a_3 t^3 + \dots,$$

on obtiendra les relations :

$$a_1 = \mathrm{M}_1,$$
$$a_2 = \mathrm{M}_2 + \mathrm{P} a_1^2,$$
$$a_3 = \mathrm{M}_3 + 2\mathrm{P} a_1 a_2 + \mathrm{P}_1 a_1^2 + \mathrm{Q} a_1^3,$$

qui de proche en proche donnent les quantités a_1, a_2, a_3, etc., en fonctions entières et à coefficients entiers de M_1, M_2, ... P, P_1, P_2, etc. Nous démontrons immédiatement ainsi le résultat découvert par M. Heine, que la série infinie qui satisfait à l'équation algébrique entre t et u, a tous ses coefficients entiers. Et si l'on revient aux variables x et z, on aura cette expression :

$$z = \frac{a_1}{n} x + \frac{a_2}{n^3} x^2 + \frac{a_3}{n^5} x^3 + \dots + \frac{a_i}{n^{2i-1}} x^i + \dots,$$

que je vais considérer à l'égard de la puissance fractionnaire du binôme : $(1-x)^{-\frac{m}{n}}$. Nous trouvons alors cette conséquence que :

$$\frac{\frac{m}{n}\left(\frac{m}{n}+1\right)\left(\frac{m}{n}+2\right) \dots \left(\frac{m}{n}+i-1\right)}{1 \cdot 2 \cdot 3 \dots i}$$
$$= \frac{m (m+n) (m+2n) \dots [m+(i-1) n]}{1 \cdot 2 \cdot 3 \dots i \cdot n^i} = \frac{a_i}{n^{2i-1}},$$

c'est à dire que l'expression :

$$\frac{m (m+n) (m+2n) \dots [m+(i-1) n]}{1 \cdot 2 \cdot 3 \dots i} n^{i-1}$$

est toujours un nombre entier.

Le procédé, dont je viens de faire usage, s'applique également aux

relations transcendantes. Considérons par example l'équation de Kepler:

$$y = a + x \sin y,$$

on fera, $y = a + u$, et on mettra la transformée:

$$u = x \sin (a+u),$$

ou plutôt:

$$u = x \sin a \,(1 - \tfrac{1}{2}u^2 + \tfrac{1}{24}u^4 - \ldots) + x \cos a \,(u - \tfrac{1}{6}u^3 + \tfrac{1}{120}u^5 - \ldots),$$

sous la forme suivante:

$$u\,(1 - x\cos a) = x \sin a - u^2 \frac{x \sin a}{2} - u^3 \frac{x \cos a}{6} - \ldots$$

Nous sommes ainsi amené à introduire, au lieu de x, la quantité $\frac{x \sin a}{1 - x \cos a}$; en la désignant par ξ pour un moment, l'équation devient en effet:

$$u = \xi - u^2 \frac{\xi}{2} - u^3 \frac{\xi \cot a}{6} \ldots\ldots,$$

et on tire très facilement:

$$u = \xi - \frac{1}{2}\xi^3 - \frac{\cot a}{6}\xi^4 - \ldots\ldots$$

Dans les Annales de l'Observatoire de Paris, M. Serret avait déjà fait la remarque, que la valeur très simple $u = \xi$, c'est à dire:

$$y = a + \frac{x \sin a}{1 - x \cos a}$$

donnait une solution approchée du problême de Kepler, en négligeant seulement le cube de l'excentricité.

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*On Correlative Pencils.** *By* Prof. Rudolf Sturm.

[*Read April 13th*, 1876.]

In two memoirs which have appeared in the "Mathematische Annalen," Vols I. and VI., I had occasion to treat the problems of homography (Projectivität) in a plane and in space. The former problem was this:—Given, in the same or in two planes, two groups of corresponding points; to find pairs of associated centres from which these groups may be projected by homographic plane pencils of rays. In the latter problem the two groups of points were situated in two spaces, and pairs of associated axes were found, from which the groups could be projected by homographic pencils of planes. In Vol. X. of the same journal I have also published a paper on the homography

* By a pencil is here meant a twofold system of lines (or planes) passing through a point (centre). When necessary, the special pencils to which the term has been usually applied will be distinguished as "plane pencils of lines (rays)" and "axial pencils of planes."

(Collineation) of pencils; the problem there being to find pairs of associated centres, from which the two groups of the two spaces may be projected by homographic pencils of rays. The present paper originated from a proposition of my friend, Dr. Hirst, whose own investigations on the correlation of two planes and of two spaces* are closely connected with mine, that I should treat the problem of the correlation (Reciprocität) of pencils in a somewhat similar, though more general manner.

1. *In one of two spaces* **A**, *suppose* k *points* A_i, l *right lines* a_i, *and* m *points* $\mathfrak{A}_i$ *to be given; and corresponding to them, in a second space* **B**, k *right lines* b_i, l *points* B_i, *and* m *points* $\mathfrak{B}_i$. I term, with Dr. Hirst, $[klm]$ the signature of the system thus defined.

Two points A, B *in the two spaces* **A**, **B** *will be termed associated points, if there be a correlation between the pencils of which they are the centres, such that* (1) *to the rays* AA_i *correspond the planes* Bb_i; (2) *to the planes* Aa_i *the rays* BB_i; *and* (3) *of two rays* $A\mathfrak{A}_i$, $B\mathfrak{B}_i$, *each lies in the plane corresponding to the other,—in other words,* $A\mathfrak{A}_i$, $B\mathfrak{B}_i$ *are conjugate rays.*

The conditions (1) and (2) are double, while (3) is a simple one; $\mathfrak{A}_i$, $\mathfrak{B}_i$ may be called a pair of conjugate points. Dr. Hirst, in his investigations, also considers conjugate lines $\mathfrak{a}_i$ $\mathfrak{b}_i$; I have not done so in this paper, since I observed that the problem became thereby much more complicated, and that the results ceased to be homogeneous.

If two points $\mathfrak{A}_i$ or $\mathfrak{B}_i$ coincide, we have a double condition (1) or (2), instead of two simple conditions (3); the identity of two lines $\mathfrak{a}_i$ or $\mathfrak{b}_i$ would not have the same effect.

2. *If* $\sigma = 2k+2l+m = 8$, *any point* A *in* **A** *may be associated with any point* B *in* **B**, *because there is one, and generally only one, correlation between the pencils of which they are the centres;* as we can see in Dr. Hirst's memoir on the correlation of two planes, No. 42. *But the signature being*

$$[220] \qquad \begin{matrix} A_1 & A_2 & a_1 & a_2 \\ b_1 & b_2 & B_1 & B_2 \end{matrix}$$

an exception occurs, whose influence will be traceable throughout our whole investigation. Two arbitrary points A, B cannot generally be associated, because the correlation [220] has no solution. With each point B, however, any point of a quadric surface α^2, passing through $A_1A_2a_1a_2$, is associated, whereof the points A are such that, a' and b' being the right lines $\overline{Aa_1a_2}$, $\overline{Bb_1b_2}$, the axial pencils of planes a' $(A_1\,A_2\,a_1\,a_2)$ and b' $(b_1\,b_2\,B_1\,B_2)$ are homographic. Between the pencil B, however, and that whose centre is at any point A of its associated

* Proc. of the Lond. Math. Soc., Vol. V., p. 40, and Vol. VI., p. 7; also Ann. di Matematica, ser. ii., vol. vi., p. 260.

surface a^2, not merely one, but a singly infinite number of correlations [220] exist; or, as we may say, there is a system of correlations [211] or [121]. Hence it is that, on adding any pair of conjugate points, $\mathfrak{A}_1\mathfrak{B}_1$, the points A and B remain associated centres. If B be moved on a certain quadric β^2, passing through $b_1\,b_2\,B_1\,B_2$, the associated surface a^2 will not change. Between the one-fold systems of quadrics $(A_1A_2a_1a_2)$ and $(b_1b_2B_1B_2)$ there is thus a (1, 1) correspondence established.

3. It is obvious that, in the cases $\sigma = 9, 10, 11$, respectively, we have, associated with each point, a doubly infinite number (surface), a singly infinite number (curve), and a finite number of points. When $\sigma > 11$, an arbitrary point has in general no associated points; but in the cases $\sigma = 12, 13, 14$, respectively, we have a surface, a curve, and a finite number of points that possess associated ones. Therefore a surface, a curve, will be associated, when $\sigma = 10, 11$, with a right line; and when $\sigma = 11, 12$, with a plane.

4. We can easily find *the degrees of multiplicity of the points* $\mathfrak{A}_i$ *and of the lines* a_i *on each of the four surfaces which, when* $\sigma = 9, 10, 11, 12$, *are respectively associated with a point* B, *a right line b, a plane* β, *the entire space* **B**.

As to the multiplicity of a point $\mathfrak{A}_i$, its degree, if $\sigma = 9$, is equal to the number of correlations $[k, l, m-1]$, where the pair $\mathfrak{A}_i\,\mathfrak{B}_i$ is withdrawn, and $\mathfrak{A}_i$, B are the associated centres (see Hirst, No. 42); if $\sigma = 10, 11, 12$, respectively, this degree is the same as the order of the surface, or the order of the curve, or the number of points associated with the point $\mathfrak{A}_i$ for the signature $[k, l, m-1]$, or, interchanging the spaces, with a point B, for $[l, k, m-1]$.

Let $\mathfrak{A}_i'$ be situated on a_i, and $\mathfrak{B}_i'$ be identical with B_i; the points B, associated with a point A on a_i, for $[klm]$, are the same as those associated with it for $[k, l-1, m+1]$, where a_i, B_i are replaced by $\mathfrak{A}_i'\,\mathfrak{B}_i'$. Therefore the degrees of multiplicity of a line a_i on our several surfaces are equal, respectively, to the number of correlations $[k, l-1, m+1]$ or $[l-1, k, m+1]$, the associated centres being arbitrary, to the order of the surface, to the order of the curve, and to the number of points associated with a point B for $[l-1, k, m+1]$.

The degree of multiplicity of the points A_i cannot be determined in this manner.

We find, however, that *the degrees of multiplicity of a point* A_i *on the curves which, when* $\sigma = 10, 11, 12, 13$, *are associated respectively with* B, b, β, **B**, are equal to the number of correlations $[k-1, l, m]$ or $[l, k-1, m]$, the associated centres being fixed, to the order of the surface, to the order of the curve, and to the number of points associated with a point B for the signature $[l, k-1, m]$.

The curves in question have as many *intersections with a line* a_i as

this line has with the surfaces respectively associated with B, b, β, $\mathbf{B}$ for $[k, l-1, m+1]$, exclusive of a point $\mathfrak{A}_i$, taken on a_i.

The curves now under consideration do not pass through the points $\mathfrak{A}_i$.

5. Two pencils whose centres A′ and A″ are associated with the same point B are homographic, and consequently generate in general a twisted cubic curve which passes through A′, A″, and the k point A_i, and cuts each of the l lines a_i twice; the planes of the pencils A′, A″, which correspond to the same ray, $B\mathfrak{B}_i$, intersect in the chord of the cubic which passes through $\mathfrak{A}_i$; hence it follows that every point A of this cubic is associated with B. If there be another point A‴ associated with B, which is not situated on this cubic, the three homographic pencils A′, A″, A‴ will generate a cubic surface, which has the points A_i for nodes, contains the lines a_i, and passes through the points $\mathfrak{A}_i$. Every point of this surface is associated with B. (Reye, "Geometrie der Lage," II., 21. Vortrag.)

If a point A, exterior to this cubic surface, were also associated with B, it would, with any two points on the surface, give rise to a new cubic surface; and thus all space would be exhausted.

Hence we infer that, *for* $\sigma = 9$, 10, 11, *a cubic surface, a* (*twisted*) *cubic curve, and a single point are respectively associated with each point* B.*

6. *When* $\sigma = 9$, *the locus of the point* A, *associated with a point* B, *has just been found to be a cubic surface* a^3, *on which the points* A_i *are double, and the lines* a_i *and the points* $\mathfrak{A}_i$ *single.* It obviously contains also the lines joining two points A_i, the transversals of four lines a_i, and the lines from each point A_i which cut two lines a_i; we shall denote these lines by $(2A_i)$, $(4a_i)$, $(A_i, 2a_i)$ respectively.

The correlation of the two pencils whose centres are B and any point A on one of these lines is an exceptional one, having the line as singular axis in A.† There are, on the whole, 6 right lines a' on a^3, the several points of which are associated with B by an exceptional correlation with singular axes, one of which is a'. To these lines belong $(2A_i)$, $(4a_i)$, $(A_i, 2a_i)$.

If B lie in a plane $(b_i\,\mathfrak{B}_k)$, the line $A_i\,\mathfrak{A}_k$ lies entirely on the corresponding surface, in all cases except the one whose signature is [221].

It is worth noting that, for the signature

$$[401] \qquad \begin{matrix} A_1\,A_2\,A_3\,A_4\,\mathfrak{A}_1 \\ b_1\;b_2\;b_3\;b_4\;\mathfrak{B}_1 \end{matrix},$$

* The same results are obtained in the corresponding problem of homography (Collineation).

† For exceptional correlations, see Hirst (l. c., especially Vol. V., Nos. 12–17), who first examined them exhaustively.

the surface associated with any point A on the plane $\alpha_{123} = A_1 A_2 A_3$ is made up of the plane $b_4 \mathfrak{B}_1$ (which arises from exceptional correlations with singular planes) and the hyperboloid $\beta^2_{123} = [b_1\, b_2\, b_3]$; whilst with each point B on this hyperboloid a cubic is associated, which breaks up into the plane α_{123}, and a quadric cone with A_4 as vertex; the latter proceeding from exceptional correlations with singular axes.

7. If b' be any ray of the pencil B, the planes corresponding to it in the several pencils A, whose centres are associated with B, all pass through the same point A′ of the cubic surface α^3 (see Reye, l. c.); this surface thus becomes a second locus of points, and between the latter A′ and the rays b' of the pencil B a (1, 1) correspondence exists.

8. The signature being

$$[311] \qquad \begin{matrix} A_1\, A_2\, A_3\, a_1\, \mathfrak{A}_1 \\ b_1\ \ b_2\ \ b_3 B_1 \mathfrak{B}_1 \end{matrix}$$

or

$$[221] \qquad \begin{matrix} A_1\, A_2\, a_1\, a_2\, \mathfrak{A}_1 \\ b_1\ \ b_2 B_1 B_2 \mathfrak{B}_1' \end{matrix}$$

the cubic surface breaks up, for an arbitrary position of B, *into the plane* $A_1 A_2 A_3$, *or* $A_1 A_2 \mathfrak{A}_1$, *and a quadric surface which passes through the points* A_i *but does not contain the lines* $(2A_i)$; *moreover, for* [221] *this quadric does not pass through the point* $\mathfrak{A}_1$. The two loci, for both the signatures, are thus distinguishable: the plane is the locus of the points A′ that correspond to the rays of B (No. 7); *the quadric is the locus of the centres* A, *which are associated with* B. For [311], the quadric associated with each point B of the plane $b_3 B_1$ degenerates into the planes $A_3 a_1$ and $A_1 A_2 \mathfrak{A}_1$, the latter of which proceeds from exceptional correlations with singular planes; while the cubic surface associated with every point A of the plane $A_3 a_1$, consists of the plane $b_3 B_1$ and a quadric, which proceeds from exceptional correlations with singular axes.

The quadric associated with B, for [221], is the same as that associated therewith for [220] (No. 2); and is, therefore, independent of $\mathfrak{A}_1 \mathfrak{B}_1$. The points of its intersection with the plane $A_1 A_2 \mathfrak{A}_1$ are, with one exception, associated with B by exceptional correlations with singular planes; the exceptional point A* is that from which A_1, A_2, the traces of a_1, a_2, and $\mathfrak{A}_1$ are projected by a plane pencil, homographic to the axial pencil of planes $(B\, b_1\, b_2)$ $(b_1, b_2, B_1, B_2, \mathfrak{B}_1)$.

For both signatures the quadric surfaces each contain three right lines a' such as those alluded to in No. 6.

9. Returning to $\sigma = 8$, I consider the signatures where $k + l = 4$; and, in the first place, [400], [130], [040], to which systems I add a pair of conjugate points $\mathfrak{A}_1\, \mathfrak{B}_1$. Let B be a fixed point, but let A move on a right line a. The plane of the pencil B, which corresponds to the several rays $A\mathfrak{A}_1$, envelopes a cone of the third class, since there are

three points on a for which it passes through $B\mathfrak{B}_1$. This cone, being unicursal, in virtue of its point by plane relation to the line a, has one double tangent plane; therefore two points lie on a, from which the points A_i and $\mathfrak{A}_1$ and the lines a_i are projected by homographic pencils; hence one twisted cubic curve exists, which passes through the $k+1$ points $A_i, \mathfrak{A}_1$, and cuts each of the $l+1$ lines a_i, a twice.

10. Let two points $\mathfrak{A}_1, \mathfrak{A}_2$ be taken on a right line $\mathfrak{a}$ in the space **A**. Each of them gives rise to a cone of the third class in the pencil B. The intersections of the tangent planes of these two cones, which correspond to the same position of A on a; in other words, the rays corresponding to the several planes $A\mathfrak{a}$, generate a unicursal cone of the sixth order. For [400] and [130] this sextic degenerates. For [400], the correlation becomes exceptional, when A comes to lie in one of the four planes such as α_{123}; this plane, in fact, and Bb_4 are the singular ones of the correlation, and the planes corresponding to $A\mathfrak{A}_1$, $A\mathfrak{A}_2$ coincide in Bb_4. The cone, therefore, becomes a quadric cone, and possesses no double edge. For [130] the correlation becomes exceptional, when A comes to lie in one of the three planes, such as A_1a_1; the plane is a singular one in the pencil A, that in B being $B\,B_2\,B_3$. We thus obtain a unicursal cone of the third order with one double edge.

For [040] the order of the cone does not diminish, but the correlation becomes exceptional whenever A falls on one of the eight transversals of a and three of the four a_i. Each of the transversals, say of $a\,a_1\,a_2\,a_3$, and BB_4 are singular axes, and therefore BB_4 corresponds to $A\mathfrak{a}$ for two positions of A. Since the four lines $B\,(B_1, B_2, B_3, B_4)$ are four of the ten double edges of our unicursal cone of the sixth order, there are six others.

There being 0, 1, 6 double edges respectively in the three cases above considered, we infer the existence of 0, 1, 6 cubics, which, in these three cases, pass through the k points A_i and cut twice the $l+2$ lines $a_i, \mathfrak{a}, a$.

11. For the signature [310], the cone of the pencil B, whose tangent planes correspond to the rays $A\mathfrak{A}_1$, A moving on a, is of the second class (No. 8), and has no double tangent plane. Hence we infer the non-existence of a cubic which passes through four points and cuts twice each of two right lines. The two cones corresponding to the rays $A\mathfrak{A}_1$, $A\mathfrak{A}_2$ generate a cone of the third order only, with one double edge; since the correlation becomes exceptional when A lies in α_{123}. Hence there is one cubic which passes through three points, and cuts each of three right lines twice.

The signature [220] alone remains to be considered. There are only two points A on the line a, between whose pencils and that of B

correlation is possible; but, then, a system of correlations exist, as if the signature were [211] or [121] (No. 2). The planes of B, corresponding to the two $A\mathfrak{A}_1$, constitute two axial pencils of planes, and the rays of B, corresponding to the two $A\mathfrak{a}$, two plane pencils of rays. (See Hirst, l. c., No. 41.) These pencils having, in the one case, a common plane, and, in the other, a common ray, we infer that one cubic exists which passes through three points, $A_1 A_2 \mathfrak{A}_1$, and meets twice three lines $a_1 a_2 a$; and also that there is one cubic which passes through two points $A_1 A_2$, and cuts twice each of four lines $a_1 a_2 \mathfrak{a}\mathfrak{a}$.

Thus the known theorems* on *cubics which pass through given points and have given right lines for chords*, are once more demonstrated, and that by means of the generation of cubics by homographic pencils.

12. We arrive at *the case*, $\sigma = 10$. *The cubic curve, which is in general associated with each point* B (A) (No. 5), *passes through the points* A_i (B_i), *and cuts each of the lines* a_i (b_i) *twice.*

If $m = 0$, the cubics $(kA_i,\ la_i)$ passing through the k points A_i and cutting twice each of the l lines a_i, are doubly infinite in number, and thus form a twofold system; the same applies to the cubics $(kb_i,\ lB_i)$. *Between these two systems of cubics* a (1, 1) *correspondence exists; each point of either space determines one cubic, and with every point of this curve the same cubic of the other system is associated.*

13. An exception† occurs for the signature

$$[320] \qquad \begin{matrix} A_1 & A_2 & A_3 & a_1 & a_2 \\ b_1 & b_2 & b_3 & B_1 & B_2 \end{matrix}.$$

All the cubics of the system $(A_1 A_2 A_3 a_1 a_2)$ are situated on the quadric surface a_0^2, which passes through $A_1 A_2 A_3 a_1 a_2$. But though no cubic passes through a point A exterior to this surface, through each point of the latter a singly infinite number of cubics, forming a one-fold system, pass.

With every point A of the first space is associated, as before, a cubic curve in the second space. When the point A moves on a right line which cuts $a_1 a_2$, the section of the pencil A $(A_1 A_2 A_3 a_1 a_2)$ made by the plane a_{123} does not change. Therefore all points A of this straight line are associated with the same point B, as well as with all other points of the cubic determined by B. *With each cubic of the second system is thus associated a right line of the congruence* $[a_1 a_2]$. *The cubic* b_9^3 *of the second system, however, which is associated with any point of the quadric* a_0^2, *is likewise associated with all other points of this surface.*

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* Cremona, Crelle's Journal, Vol. lx., p. 188; Sturm, *ibidem*, Vol. lxxx., p. 128.

† Dr. Hirst pointed out this exception in a letter to me, written in April, 1875.

14. If $m > 0$, no two points have, in general, the same associated cubic.

The signature being

$$[222] \qquad \begin{matrix} A_1\, A_2\, a_1\, a_2\, \mathfrak{A}_1\, \mathfrak{A}_2 \\ b_1\, b_2\, B_1 B_1\, \mathfrak{B}_1\, \mathfrak{B}_2 \end{matrix},$$

the third intersection with the plane $A_1 A_2 \mathfrak{A}_1$, of the cubic associated with any point B, does not depend on the pair $\mathfrak{A}_2 \mathfrak{B}_2$; it is the point A* referred to in No. 8.

15. To any system $[klm]$, where $\sigma = 8$, add $\mathfrak{A}_1 \mathfrak{B}_1$ and $\mathfrak{A}_2 \mathfrak{B}_2$. The two cubic surfaces associated with any point B, for $\left(8, \begin{matrix}\mathfrak{A}_1 \\ \mathfrak{B}_1\end{matrix}\right)$ and $\left(8, \begin{matrix}\mathfrak{A}_2 \\ \mathfrak{B}_2\end{matrix}\right)$, intersect in the cubic curve associated with B for $\left(8, \begin{matrix}\mathfrak{A}_1\, \mathfrak{A}_2 \\ \mathfrak{B}_1\, \mathfrak{B}_2\end{matrix}\right)$; the remaining intersection contains the l lines a_i, and all lines $(2A_i)$, $(4a_i)$, $(A_i, 2a_i)$. Therefore the residual curve (which does not exist for [130], [400], or [040]) is, for [008] and [106], of the 6th order; for [016], [114], and [204], of the 5th order; for [212] and [024], of the 4th order; and for [122], [302], and [032], of the 3rd order.

If A *be a point on this curve of the sixth order, the correlation between the pencils* A *and* B *is not determined by the eight conditions; there is, in fact, a system of such correlations.* If A be situated on a line $(2A_i)$, $(4a_i)$, $(A_i, 2a_i)$, these correlations are all exceptional ones with singular axes; but if A lie on a line a_i, or on the residual curve, the correlations are, in general, of the ordinary kind.

16. The signature being

$$[212] \qquad \begin{matrix} A_1\, A_1\, a_1\, \mathfrak{A}_1\, \mathfrak{A}_2 \\ b_1\, b_2\, B_1\, \mathfrak{B}_1\, \mathfrak{B}_2 \end{matrix},$$

this residual curve (of the fourth order) consists of two conics, situated in the planes $A_1 A_2 \mathfrak{A}_1$, $A_1 A_2 \mathfrak{A}_2$, and respectively independent of $\mathfrak{A}_2 \mathfrak{B}_2$, $\mathfrak{A}_1 \mathfrak{B}_1$.

The signature being

$$[122] \qquad \begin{matrix} A_1\, a_1\, a_2\, \mathfrak{A}_1\, \mathfrak{A}_2 \\ b_1\, B_1 B_2\, \mathfrak{B}_1\, \mathfrak{B}_2 \end{matrix},$$

it is a cubic situated in the plane $A_1 \mathfrak{A}_1 \mathfrak{A}_2$, and passing twice through A_1.

For the two signatures [302] and [032], it is a twisted cubic, and we can find two straight lines $\mathfrak{b}_1, \mathfrak{b}_2$ passing through $\mathfrak{B}_1$, $\mathfrak{B}_2$ respectively, such that the cubic is associated with B for $\left(300, \begin{matrix}\mathfrak{A}_1\, \mathfrak{A}_1 \\ \mathfrak{b}_1\, \mathfrak{b}_2\end{matrix}\right)$ and $\left(030, \begin{matrix}\mathfrak{A}_1\, \mathfrak{A}_2 \\ \mathfrak{b}_1\, \mathfrak{b}_2\end{matrix}\right)$.

17. Hitherto the two signatures [310] and [220] have been omitted. The two quadric surfaces associated with B for $\left(310, \begin{matrix}\mathfrak{A}_1 \\ \mathfrak{B}_1\end{matrix}\right)$ and

$\left(310, \frac{\mathfrak{A}_2}{\mathfrak{B}_2}\right)$ (No. 8) intersect in the one line a_i, and the cubic associated with B for $\left(310, \frac{\mathfrak{A}_1\,\mathfrak{A}_2}{\mathfrak{B}_1\,\mathfrak{B}_2}\right)$; there is no residual curve. For the other signature the two quadrics are identical; instead of a curve we have now a quadric surface, each of whose points A is associated with B by a one-fold system of ordinary correlations (No. 2).

18. *If* B *move on a right line* b, *the associated cubic curve will generate a surface;* and it is obvious, *that the surface in the second space, similarly associated with a right line* a *of the first, will be of the same order.*

Generally, this order is 10; *and on the surface* α^{10} *in the first space the points* A_i *are sextuple, the lines* a_i *and the points* $\mathfrak{A}_i$ *being triple.* There are 20 right lines a' on the surface α^{10}, such that all points of any one of them are associated with the same point B of b, by an exceptional correlation with singular axes, one of which is a'; the lines $(2A_i)$, $(4a_i)$ and $(A_i, 2a_i)$ each count as two among these 20 lines, and are therefore double lines of the surface α^{10}.

19. *But now we have a greater number of exceptions.* In the following table of them, n is the order of the surface, and s the number of singular axes; the other columns give the degrees of multiplicity of the elements placed at their head.

Signature.	n	s	A_i	a_i	$\mathfrak{A}_i$	$(2A_i)$	$(4a_i)$	$(A_i, 2a_i)$
410	6	8	3	3	...	0	...	...
140	6	8	2	2	...	...	2	0
320	6	8	2	3	...	0	...	2
230	6	12	3	2	...	0	...	1
312	9	17	5	3	3	1	...	...
132	9	17	5	3	2	...	...	2
222	8	14	4	3	2	0	...	2

The surface of the sixth order, for [320], is a ruled one; since to a point B corresponds a right line, not a cubic (No. 13).

On the surface of the sixth order, for [230], the cubic curve a_0^3, which corresponds to the whole surface β_0^2 (No. 13), is double.

20. For the values 0 and 2 of m, I have found the order of the surface, and the degrees of multiplicity of A_i, a_i, $\mathfrak{A}_i$ [the two latter of which are already known (No. 4)], by considering the correspondence between those points, on the line b, which are associated with the same point of a line a in two systems $[k, l, m-1]$, which differ from one another only by one pair of conjugate points. If there be not two such pairs in the original system $[klm]$, a pair $\frac{A_i}{b_i}$, or $\frac{a_i}{B_i}$, is to be broken up into $\frac{\mathfrak{A}_i\,\mathfrak{A}_i}{\mathfrak{B}_i\,\mathfrak{B}_k}$,

or into $\begin{matrix}\mathfrak{A}_i\,\mathfrak{A}_k\\ \mathfrak{B}_i\,\mathfrak{B}_i\end{matrix}$. The above exceptions arise, either from the identity of two points $\mathfrak{A}_i$ or $\mathfrak{B}_i$, or from the circumstance, that for some signatures $[k,\ l,\ m-1]$, the surface associated with B is a quadric, instead of a cubic. In determining the degrees of multiplicity of A_i, a_i, and $\mathfrak{A}_i$, the line a must be supposed to pass through an A_i, to meet an a_i, or to pass through an $\mathfrak{A}_i$.

By supposing the line b to pass through a point B_i, or $\mathfrak{B}_i$, or to cut a line b_i, we find the reduction in the order of the associated surface, and in the degrees of multiplicity of A_i, a_i, $\mathfrak{A}_i$ thereon.

With the signature [222] the correspondence above described could not be made use of, for the points on a and b, which are associated for $\begin{matrix}A_1\,A_2\,a_1\,a_2\,\mathfrak{A}_1\\ b_1\,b_2\,B_1B_2\mathfrak{B}_1\end{matrix}$, remain associated when $\mathfrak{A}_1\,\mathfrak{B}_1$ is replaced by another pair.

For this and the other signatures I employed a procedure of general character, applicable to the cases $\sigma = 9, 10, 11, 12$, and by means of which we can pass readily, from signatures that have been already submitted to ordinary treatment, to others less susceptible of being so. For all signatures the number s, and the degrees of multiplicity of $(2A_i)$, $(4a_i)$, $(A_i, 2a_i)$ could be found by theorems contained in, or easily deduced from, my paper on homography in space (Math. Annalen, vol. vi.), because an exceptional correlation with singular axes implies the homography of axial pencils of planes.

21. When $\sigma=10$, with a point B_i (A_i) each point of the other space is associated; with a point $\mathfrak{B}_i(\mathfrak{A}_i)$ or a point of $b_i(a_i)$ is associated, instead of a curve, an entire surface, generally of the third order (see No. 4). Besides these, however, there are other points, with which a surface is associated; we shall find them on considering the case $\sigma=11$.

22. *In this case,* $\sigma = 11$, *with each point* B *or* A, *one, and only one point* A *or* B *in the other space is associated*, as we have already found (No. 5), or as we can see in the following manner: Consider the case $\left(10, \begin{matrix}\mathfrak{A}_1\\ \mathfrak{B}_1\end{matrix}\right)$. For (10), in general, a cubic curve is associated with each point B; the planes corresponding to $B\,\mathfrak{B}_1$ in the several pencils whose centres A lie on this curve pass through a fixed chord thereof; one of these planes therefore, for a certain position of A on the curve, passes through $\mathfrak{A}_1$.

If the signature be

$$[321] \qquad \begin{matrix}A_1\,A_2\,A_3\,a_1\,a_2\,\mathfrak{A}_1\\ b_1\ \ b_2\ \ b_3\,B_1B_2\mathfrak{B}_1\end{matrix},$$

then since, for [320], a right line a^1, cutting $a_1\,a_2$, is associated with each point B (No. 13), the plane corresponding to $B\,\mathfrak{B}_1$ passes, for all positions of A on a^1, through the same right line of the plane a_{123}, the intersection of this plane with the several pencils being invariable.

23. Consequently *we have, for all signatures, a point by point correspondence between the two spaces.* It is known that the order of the curve in either space which is associated with a right line of the other, is equal to the order of the surface in the latter space, which is associated with a plane of the former.

For all signatures, [321] *excepted, a surface of the eleventh order is associated with each plane* β (α), *on which surface the points* A_i (B_i) *are sextuple, the lines* a_i (b_i) *and the points* $\mathfrak{A}_i$ ($\mathfrak{B}_i$) *being triple* (see No. 4). *Both transformations, therefore, are of the same order. The curve associated with a right line* b (a) *is likewise of the eleventh order; it passes three times through each* A_i (B_i), *and meets each* a_i (b_i) ***seven times*** (see No. 4).

In order to find these results for certain signatures, when $m=1$ or 3, $(11)=\left(8, \begin{smallmatrix}\mathfrak{A}_1 & \mathfrak{A}_2 & \mathfrak{A}_3 \\ \mathfrak{B}_1 & \mathfrak{B}_2 & \mathfrak{B}_3\end{smallmatrix}\right)$ was resolved into $(10)=\left(8, \begin{smallmatrix}\mathfrak{A}_2 & \mathfrak{A}_3 \\ \mathfrak{B}_2 & \mathfrak{B}_3\end{smallmatrix}\right)$, and $(9)=\left(8, \begin{smallmatrix}\mathfrak{A}_1 \\ \mathfrak{B}_1\end{smallmatrix}\right)$; after breaking up, when necessary, a double condition into two single ones, as above described. Let a be a right line in the first space, β a plane in the second; the surface, generally of 10th order, associated with a for (10), is cut by β in a curve of the same order. On this curve we have a correspondence between those points which are associated with the same point A of a for (10) as well as for (9). The Cayley-Brill theorem* gives the number of coincidences; the superfluous ones among them, however, cause some difficulty. By passing a through A_i, we find the degree of multiplicity of A_i on the surface α^{11}.

The other signatures, as in the case $\sigma=10$, were treated by the procedure alluded to in No. 20.

24. By employing theorems given in my paper on homography in space (Math. Ann., vol. vi.), I have found, that *there are, for all signatures here under consideration, twenty pairs of right lines* a', b' *such that by an exceptional correlation, of which* a', b' *are the singular axes, each point of either of these lines is associated with every point of the other.* Since in a correlation with singular axes the pencils of planes around these axes are necessarily homographic, it is obvious that, if the vertex be moved on such an axis, the correlation does not cease to exist.

Thus our twenty right lines a' (or b') form those fundamental curves of the homaloidal system of surfaces in the space **A** (or **B**), of which Cremona speaks in his memoir *Sulle trasformazioni razionali nello spazio* (Annali di Matematica, ser. ii., T. v., p. 131, No. 7).

25. Since each plane β (α) cuts each of the lines b' (a') once, *the lines* a' (b') *are situated singly on the associated surfaces* α^{11} (β^{11}). Two sur-

* *Comptes Rendus,* Vol. 62, p. 586; *Math. Ann.,* Vol. vi., p. 33, Vol. vii., p. 607.

faces a^{11} intersect in the curve a^{11} associated with the intersection of their associated planes β, in the 20 lines a', and consequently also in a curve of 90th order. Each point of this curve has an associated point in each of the two planes β, but not in their intersection; consequently it is associated with an entire cubic curve (No. 5); and since this cubic meets each plane β three times, the curve of the 90th order resolves itself into *a curve of the 10th order which is triple on all surfaces of the homaloidal system in the space* **A**. This is the fundamental curve of the system treated by Cremona (l. c., No. 6.)

Thus there are in each space a singly infinite number of points, with each of which, not a single point merely, but a whole cubic curve is associated. These points lie on a curve of the 10th order a_0^{10} (b_0^{10}), *to which evidently belong the lines* a_i (b_i). *This curve, or its residual part, passes three times through each of the points* A_i (B_i), *once through each of the points* $\mathfrak{A}_i$ $(\mathfrak{B}_i)$, *and cuts three times each of the lines* a_i (b_i).

26. With a point B_i, say B_1, is associated, when $\sigma = 11$, a cubic surface which is the same as the associate of B_1, for $\left(11, -\frac{a_1}{B_1}\right)$; viz., when the pair $a_1 B_1$ is withdrawn. Accordingly, with a plane β that passes through B_1, a surface is associated which breaks up into the cubic surface, thus associated with B_1, and another surface a^8 of the 8th order. Six of the 20 lines a' are situated on the cubic surface (No. 6); the remaining 14 are on the residual surface a^8. The curve a_0^{10} is composed of a_1, and a curve of the 9th order; the cubics associated with the points of the latter cut β in B_1, and in two other points; but those with which the points of a_1 are associated do not pass through B_1. Therefore the curve of the 9th order is double on a^8 and single on the cubic surface associated with B_1; a_1 is triple on the former. The intersection of this cubic surface and the surface a^{11} associated with an arbitrary plane β, consists of the six lines a' and the triple curve of the 9th order, therefore of fundamental curves solely (Cremona, l. c. No. 17).

27. The cubics associated with the several points of a line b_i, say b_1, generate a surface of the 7th order; since there are, on a right line a, 7 points which have associated ones on b_1 (No. 23) for all signatures now under consideration. The corresponding point A_1 is triple on this surface.

28. The cubics associated with the points of the residual curve, obtained on subtracting from b_0^{10} the lines b_i, are those which are associated therewith for $[k, l, m-1]$, one pair $\mathfrak{A}_i\mathfrak{B}_i$ being withdrawn.

The order of the surface generated by these cubics is equal to the number of those intersections of the residual curve with the surface

associated, for $[k, l, m-1]$, with a right line a, which actually possess an associated cubic curve (the same as for $[klm]$) and not a surface.

The surface whose order is thus determined, the surfaces which proceed from the lines b_i (No. 27), and the cubic surfaces associated with the points B_i, the last counted as double (Cremona, No. 13), constitute *the Jacobian of the homaloidal system in the space* **A**. This Jacobian is known to be a surface of the 40th order (Cremona, No. 6).

29. Let us take, as an example,

$$[215] \qquad \begin{matrix} A_1 & A_2 & a_1 & \mathfrak{A}_1 \dots \mathfrak{A}_5 \\ b_1 & b_2 & B_1 & \mathfrak{B}_1 \dots \mathfrak{B}_5 \end{matrix}.$$

We have one (double) cubic surface, the associate of B_1, and two surfaces of the seventh order, associated with b_1, b_2. The remaining part of the Jacobian is of the 20th order. The curve b_0^{10} breaks up into b_1, b_2, and a curve b^8 of the 8th order, which cuts the surface β^{10} associated with any right line a, for

$$[214] \qquad \begin{matrix} A_1 & A_2 & a_1 & \mathfrak{A}_1 \dots \mathfrak{B}_4 \\ b_1 & b_2 & B_1 & \mathfrak{B}_1 \dots \mathfrak{B}_4 \end{matrix},$$

in B_1, $\mathfrak{B}_1$, ..., $\mathfrak{B}_4$, in points on b_1, b_2, and likewise in 32 points, amongst which are the 20 associated with the intersections of a and the remaining part of the Jacobian above referred to.

Each of the remaining 12 points must, for [214], possess an associated point on a which is not situated on the cubic curve associated therewith for [215] as well as for [214]. It must therefore have an associated cubic surface (No. 5). As this cubic surface cuts the line a three times, each of these points is triple on β^{10}; consequently their number is $\frac{1}{3} . 12 = 4$.

30. We will consider another example:

$$[313] \qquad \begin{matrix} A_1 & A_2 & A_3 & a_1 & \mathfrak{A}_1 & \mathfrak{A}_2 & \mathfrak{A}_3 \\ b_1 & b_2 & b_3 & B_1 & \mathfrak{B}_1 & \mathfrak{B}_2 & \mathfrak{B}_3 \end{matrix}.$$

Three surfaces of the 7th order and a double cubic surface being subtracted, the remaining part of the Jacobian is of the 13th order. The residual part of b_0^{10} is of the 7th order and cuts the surface β^8 associated with the line a for $[312] = \left[313, -\frac{\mathfrak{A}_3}{\mathfrak{B}_3}\right]$ in 17 points, exclusive of B_1, $\mathfrak{B}_1$, $\mathfrak{B}_2$, and those on b_1, b_2, b_3. Hence we infer that there are $\frac{1}{2}(17 - 13) = 2$ points, with which, for [312] as well as for $[311] = \left[312, -\frac{\mathfrak{A}_2}{\mathfrak{B}_2}\right]$, a whole surface of the second degree is associated.

After having treated, in a similar way, the other signatures of $\sigma = 10$, the following was the result arrived at: *When* $\sigma = 10$, *there are, exclusive of the fundamental elements,* $10 - 2\,(k+l)$ *points* A_0 (B_0), *in each of the two spaces* **A** (**B**), *with each of which is associated not only a cubic*

curve but a whole surface, generally of the third order; if the signature be [312] or [222], the surfaces associated with the B_0 are quadrics.

These points $A_0\,B_0$ *are triple (resp. double) on all surfaces of their space which are associated with right lines in the other.*

If $k+l=5$, there are no such points; a result which might have been anticipated, seeing that there exists a (1,1) correspondence between the two-fold systems of cubics.

31. We proceed to the signature, hitherto excluded (No. 23); viz.

$$[321] \qquad \begin{matrix} A_1\,A_2\,A_3\,a_1\,a_2\,\mathfrak{A}_1 \\ b_1\;b_2\;b_3\,B_1\,B_2\,\mathfrak{B}_1 \end{matrix}$$

(or [231] if the spaces be interchanged). By the method explained in No. 23, it may be found that *with each plane* $\beta\,(\alpha)$ *is associated a surface of the 9th order* $\alpha^9\,(\beta^9)$, *on which the points* $A_i\,(B_i)$ *are quadruple, and the lines* $a_i\,(b_i)$ *triple; the point* $\mathfrak{A}_1$ *being triple on* α^9, *but* $\mathfrak{B}_1$ *single on* β^9. *The curve* $a^9\,(b^9)$, *therefore, which is associated with a right line* $b\,(a)$ *is also of the 9th order; it passes twice through each point* $A_i\,(B_i)$, *and cuts each line* $a_i\,(b_i)$ *six times.*

32. *There are* 14 *pairs of right lines* a', b' (No. 24); consequently the residual curve of intersection of two surfaces α^9 or β^9 is of the 58th order. In the space **A** this curve includes *a (unicursal) plane quartic* $\bar{a}^4$ *situated in* α_{123}, *independent of* $\frac{\mathfrak{A}_1}{\mathfrak{B}_1}$, *passing twice through* A_1, A_2, A_3, *and cutting* a_1, a_2. *With each point of this quartic are associated all points of a generator of the hyperboloid* β^2_{123}, of which b_1, b_2, b_3 are directors. Therefore *the curve* $\bar{a}^4$ *is single on all surfaces* α^9; the part it contributes to the Jacobian consists of the hyperboloid β^2_{123}. The remaining curve (of the 54th order) is *a sextic curve, triple on all surfaces* α^9, *and composed of* a_1, a_2, *and a twisted quartic curve* a^4_0, which passes once through $A_1A_2A_3$ and cuts a_1, a_2 twice. With each point of this remaining curve a cubic curve is associated.

33. In the space **B** the curve of the 58th order includes, likewise, *a quartic curve* $\bar{b}^4$ *situated in the plane* $B_1B_2\mathfrak{B}_1$, *passing twice through* B_1B_2, *once through* $\mathfrak{B}_1$, *and cutting* b_1, b_2, b_3. In fact, with each point B is associated, for $[320] = \left[321, -\frac{\mathfrak{A}_1}{\mathfrak{B}_1}\right]$, a right line a^1, cutting a_1, a_2. The fixed line, in the plane α_{123}, through which the plane corresponding to $B\mathfrak{B}_1$ always passes, wherever A may be on a^1 (No. 13, 22), must cut a^1, if B be situated in the plane $B_1B_2\mathfrak{B}_1$; therefore the corresponding plane is invariable. The curve $\bar{b}^4$ is generated by those points B, for which this plane passes through $\mathfrak{A}_1$; consequently a whole right line a^1 is also associated with each of these points B for [321]. *The curve* $\bar{b}^4$ *is single on all surfaces* β^9; *the remaining curve is*, as in the other space, *a sextic*

curve, triple on all surfaces β^9, *and breaking up into* b_1, b_2, b_3, *and the cubic* b_0^3, *with each of whose points the whole surface* a_0^2 *is associated*, for [320] (No. 13), and therefore a cubic curve on a_0^2 for [321].—The third point of b_0^3, in the plane $B_1B_2\mathfrak{B}_1$, is single on the quartic b^4. The other points A, B of the planes $A_1A_2A_3$, $B_1B_2\mathfrak{B}_1$ are associated, point with point, by exceptional correlations which have these planes for singular ones.

34. With each of the points A_i (B_i) is associated, for this signature, a quadric surface, on which three of the 14 lines b', (a') are situated.

The Jacobians of each space are of the thirty-second order.

That of the space **A** consists (1) of two double quadrics, proceeding from the two B_i; (2) of three surfaces of the 6th order, proceeding from the three lines b_i, and each having a double point at the corresponding A_i; (3) of the quadric a_0^2 associated with b_0^3; and (4) of a ruled surface of the fourth degree, generated by the right lines a^1 which are associated with the points of the plane quartic $\overline{b}^4$ (for the surface β^6, which corresponds to a right line a, for [320], is cut by b^1 in B_1, B_2, in the third intersection of b_0^3, in the traces of b_1, b_2, b_3, and in 4 other points).

The Jacobian of the space **B** is composed (1) of three double quadrics; (2) of two surfaces of the 6th order; (3) of the hyperboloid β_{123}^2; and (4) of another surface of the 6th order, proceeding from the twisted quartic curve a_0^4.

There are no points A_0, B_0; $k+l$ being equal to 5.

35. We have now arrived at *the case* $\sigma = 12$. *An arbitrary point has, in general, no associated point. The locus of a point* $(A)_{12}$, *which possesses an associated point* $(B)_{12}$, *is, in fact*—except for the signature [330]—*a surface of the fourth order* a^4, *on which the points* A_i *are double, and the lines* a_i *and the points* $\mathfrak{A}_i$ *single*, but which no longer contains the lines $(2A_i)$, $(4a_i)$, $(A_i, 2a_i)$. The associated points $(B)_{12}$ lie on a similar surface β^4. *For the exceptional signature* [330] *both the surfaces are of the third order, and simply pass through* A_i, B_i.* I have found these results, for those signatures in which $m \leqq 4$, by resolving $\left(8, \frac{\mathfrak{A}_1\,\mathfrak{A}_2\,\mathfrak{A}_3\,\mathfrak{A}_4}{\mathfrak{B}_1\,\mathfrak{B}_2\,\mathfrak{B}_3\,\mathfrak{B}_4}\right)$—where also the two former or the two latter $\mathfrak{A}_i$ or $\mathfrak{B}_i$ may be united—into $\left(8, \frac{\mathfrak{A}_1\,\mathfrak{A}_2}{\mathfrak{B}_1\,\mathfrak{B}_2}\right)$ and $\left(8, \frac{\mathfrak{A}_3\,\mathfrak{A}_4}{\mathfrak{B}_3\,\mathfrak{B}_4}\right)$, and considering, on a right line a, the correspondence between those points which have a common associate in both the systems thus obtained.

* This exception was likewise detected by Dr. Hirst, and communicated to me in the above quoted letter.

The other signatures were treated by the general procedure alluded to in No. 20.

36. Whatever σ may be, if a cubic curve be associated with a point, one point of this cubic will also be associated therewith for $\left(\sigma, \begin{smallmatrix}\mathfrak{A}_1\\ \mathfrak{B}_1\end{smallmatrix}\right)$ (No. 22).

Therefore *the curves of the* 10*th order*, a_0^{10}, b_0^{10}, *which were found for* $\sigma = 11$, *lie wholly as single curves on the respective surfaces* α^4, β^4 *belonging to* $\left(11, \begin{smallmatrix}\mathfrak{A}_1\\ \mathfrak{B}_1\end{smallmatrix}\right)$.

The identity of the added $\mathfrak{A}_1$ or $\mathfrak{B}_1$ with one of the $\mathfrak{A}_i$ or $\mathfrak{B}_i$ already contained in (11), by which arises a new pair $\begin{smallmatrix}A_i\\ b_i\end{smallmatrix}$ or $\begin{smallmatrix}a_i\\ B_i\end{smallmatrix}$, does not change this result.

37. The signature

$$[321] \qquad \begin{matrix}A_1\,A_2\,A_3\,a_1\,a_2\,\mathfrak{A}_1\\ b_1\;b_2\;b_3\,B_1B_2\mathfrak{B}_1\end{matrix}$$

requires special treatment.

We have in the space $\mathbf{A}(\mathbf{B})$ a plane quartic $\bar{a}^4$ ($\bar{b}^4$), with each point A (B) of which a right line b^1 (a^1) is associated (No. 32, 33). One point of this line is also associated therewith for $\left(321, \begin{smallmatrix}\mathfrak{A}_2\\ \mathfrak{B}_2\end{smallmatrix}\right)$; because, when B (A) moves on b^1 (a^1), the plane corresponding to the ray $A\mathfrak{A}_2$ ($B\mathfrak{B}_2$) rotates around a fixed line of the plane $B_1B_2\mathfrak{B}_1$ ($A_1A_2A_3$), and therefore passes once through $\mathfrak{B}_2$ ($\mathfrak{A}_2$).

Hence the two plane quartics $\bar{a}^4$, $\bar{b}^4$ lie on the surfaces α^4, β^4 respectively; as likewise do the residual curves of the 6th order; therefore altogether two curves of the tenth order are situated thereon, as for the other signatures.

38. But if to [321] be added $\begin{smallmatrix}\mathfrak{A}_2\\ \mathfrak{B}_1\end{smallmatrix}$, whereby we obtain the signature [330]; *the two plane quartics do not lie on the surface* α^3, β^3. In fact, if A be a point of $\bar{a}^4$, and b^1 the associated right line, the plane corresponding to $A\mathfrak{A}_2$ and passing through $\mathfrak{B}_2 \equiv \mathfrak{B}_1$ coincides with $B_1B_2\mathfrak{B}_1$, and the point B it determines on b^1 is $(b^1, B_1B_2\mathfrak{B}_1)$, from which the system in the plane $B_1B_2\mathfrak{B}_1$ could not be projected by any pencil correlative to the pencil A. Similarly, if B lie on $\bar{b}^4$, the plane corresponding to $B\mathfrak{B}_2 \equiv B\mathfrak{B}_1$ is the same for all positions of A on the associated right line a^1 (No. 33), and, generally, does not pass through $\mathfrak{A}_2$.

39. For the signature

$$[330] \qquad \begin{matrix}A_1\,A_2\,A_3\,a_1\,a_2\,a_3\\ b_1\;b_2\;b_3\,B_1B_2B_3\end{matrix}$$

the two cubic curves, which are the intersections of the planes $\alpha_{123} = A_1A_2A_3$ and $\beta_{123} = B_1B_2B_3$ with the surfaces α^3, β^3 respectively, are

associated, point with point, by exceptional correlations with singular planes. In each of the two planes we have, in fact, a group of six points; in the one the points A_i and the traces of the a_i, in the other the traces of the b_i and the points B_i. The two cubic curves are precisely those I have referred to in my first paper on homography (Math. Ann., vol. i.) when considering two groups of six points.

The surface α^3 intersects the hyperboloid $\alpha^2_{123} = [a_1 a_2 a_3]$, not only in $a_1 a_2 a_3$, but also in three generators a^0; with all points of such a generator, one and the same point B^0 of the curve in the plane β_{123} is associated by ordinary correlations. We have thus three points B^0 in β_{123}, and similarly three points A^0 in α_{123}.

When, for other signatures, there is such a hyperboloid α^2_{ikl}, the points of its residual intersection with α^4 are associated with those of the intersection of β^4 and the plane β_{ikl}.

40. *With the points on the intersection of an arbitrary plane β and the surface β, are associated those of the curve of the fourteenth order a^{14}*, which remains common to α^4 and to the surface α^{11}, associated with β for $\left(12, -\frac{\mathfrak{A}_1}{\mathfrak{B}_1}\right)$, after deducting the curve a^{10}_0, which is single on α^4 and triple on α^{11}. *This curve a^{14} passes three times through each point A_i, and cuts each line a_i eight times.*

For the signature

$$[322] \qquad \begin{matrix} A_1 A_2 A_3 \, a_1 \, a_2 \, \mathfrak{A}_1 \mathfrak{A}_2 \\ b_1 \, b_2 \, b_3 \, B_1 B_2 \mathfrak{B}_1 \mathfrak{B}_2 \end{matrix}$$

the surface associated with β, for $\left(322, -\frac{\mathfrak{A}_2}{\mathfrak{B}_2}\right)$, is of the 9th order; it has in common with α^4 the plane quartic $\bar{a}^4$, single on both, and the curve of the sixth order, single on α^4, but triple on α^9 (No. 32); the residual curve is of the fourteenth order.

But if $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be identical; that is to say, *if the signature be* [330], the plane quartic does not lie on α^3; *the curve associated with the intersection $\beta_i \beta^3$, or briefly with the plane β, is of the ninth order, passes once through A_1, A_2, A_3, and cuts each of the lines a_1, a_2, a_3 six times.*

41. With a point B_i a cubic curve is associated for all signatures where $\sigma = 12$, except [330], in which case the associate is a right line. With a line b_i a curve of the eighth order is associated, because there are, in any plane α, eight points, each of which has an associated one on b_i (No. 40); this curve passes twice through all points A_i which do not correspond to the line b_i under consideration. For the signature [330], the curve associated with each line b_i is of the sixth order, passes once through the non-corresponding points A_i, but not through the corresponding one.

42. When $\sigma = 10$, the points A associated with any point B lie on a

cubic curve, for all signatures except [320]. The plane of the pencil A, corresponding to the ray of B, which passes through a fixed point B′, rotates around a chord $\bar{a}$ of the cubic. I allow B to move on a right line b, and seek the order of the ruled surface generated by the chord $\bar{a}$. The number of chords $\bar{a}$ cut by an arbitrary right line a' is the same as the number of points B on b, each of which possesses an associated point for $\left(10, \overset{a'}{\mathrm{B}'}\right)$; therefore, in all cases except [320], which leads to the exception [330], the sought surface is of the fourth order. If B′ lie on b, however, it is easy to see that there are but two points $(\mathrm{B})_{12}$ on b, different from B′; and consequently that, in this case, the generated surface is a quadric.

For the signature [320], where a right line is associated with each point B, the axis $\bar{a}$ is situated in the plane α_{123}, and, the locus of the points $(\mathrm{B})_{12}$, for [330], being a cubic surface, on which B′ is single, the chords $\bar{a}$ envelop, in the general case, a plane curve of the third class, and in the particular case, a conic.

43. We pass to *the case where* $\sigma = 13$, and resolve it into $\left(11, \overset{\mathfrak{A}_1}{\mathfrak{B}_1}\right)$ and $\left(11, \overset{\mathfrak{A}_2}{\mathfrak{B}_2}\right)$. The two surfaces α^4, belonging to these two pairs of groups, intersect in the curve a_0^{10}, with each point of which, for (11), a singly infinite number of points are associated, and in *a curve of the sixth order*. Each point of this sextic curve has but one associated point for (11), and this is associated thereto, both for $\left(11, \overset{\mathfrak{A}_1}{\mathfrak{B}_1}\right)$ and for $\left(11, \overset{\mathfrak{A}_2}{\mathfrak{B}_2}\right)$, and hence also for (13).

In the case of the signature

$$[331] \qquad \begin{matrix} \mathrm{A}_1\mathrm{A}_2\mathrm{A}_3\, a_1\, a_2\, a_3\, \mathfrak{A}_1 \\ b_1\; b_2\; b_3 \mathrm{B}_1\mathrm{B}_2\mathrm{B}_3\, \mathfrak{B}_1 \end{matrix}$$

I substitute $\overset{\mathfrak{A}_2}{\mathfrak{B}_2}$, $\overset{\mathfrak{A}'_2}{\mathfrak{B}_2}$, in place of $\overset{a_3}{\mathrm{B}_3}$, $\mathfrak{A}_2$ and $\mathfrak{A}'_2$ being situated on a_3, and $\mathfrak{B}_2$ identical with B_3, and resolve into the two pairs of groups

$$\begin{matrix} \mathrm{A}_1\mathrm{A}_2\mathrm{A}_3\, a_1\, a_2\, a_3 \\ b_1\; b_2\; b_3\, \mathrm{B}_1\mathrm{B}_2\mathrm{B}_3 \end{matrix} \quad \text{and} \quad \begin{matrix} \mathrm{A}_1\mathrm{A}_2\mathrm{A}_3\, a_1\, a_2\, \mathfrak{A}_2\mathfrak{A}_1 \\ b_1\; b_2\; b_3\, \mathrm{B}_1\mathrm{B}_2\mathfrak{B}_2\mathfrak{B}_1 \end{matrix}.$$

The former pair gives rise to a surface α^3, the latter to a surface α^4; the curve of the sixth order, with each point of which a cubic is associated for

$$\begin{matrix} \mathrm{A}_1\mathrm{A}_2\mathrm{A}_3\, a_1\, a_2\, \mathfrak{A}_2 \\ b_1\; b_2\; b_3\, \mathrm{B}_1\mathrm{B}_2\mathfrak{B}_2 \end{matrix},$$

is situated on both surfaces α^3 and α^4, but not the quartic in the plane α_{123}. Therefore the residual intersection is again a curve of the sixth order. Hence we conclude that, *whatever may be the signature, when* $\sigma = 13$, *the points* $(\mathrm{A})_{13}$, *each of which possesses an associate* $(\mathrm{B})_{13}$, *lie on a curve*

of the sixth order a^6, which passes once through each point A_i, and cuts each line a_i three times.

The corresponding points $(B)_{13}$ lie on a similar curve b^6. The number of passages of the curve a_0^{10}, when $\sigma = 11$, through a point A_i, and that of its intersections with a line a_i may now be verified.*

44. The question arises: *In how many points, exclusive of the A_i, does the curve a^6 cut this curve a_0^{10}?*

The curve a^6 intersects the surface α^{11}, which is associated, for $\sigma = 11$, with a plane β, in sixty-six points. These are (1) the points A_i, (2) the six points associated with the intersections of b^6 and β, and (3) the points each of which has an associated point, for $\sigma = 11$, situated in β, as well as another, also associated therewith, for $\sigma = 13$. Consequently these are points which have, for $\sigma = 11$, an infinite number of associated points (No. 5), and are situated on a_0^{10}. The curve a_0^{10} being triple on α^{11}, each point of the third kind is a triple intersection of a^6 and α^{11}, but a single one of a^6 and a_0^{10}.

For all signatures, therefore, the number of the intersections of these two curves is $\frac{1}{3}(6 \cdot 11 - 6 - 6k) = 20 - 2k$.

45. No exception to this occurs when to the signature

$$[321] \qquad \begin{matrix} A_1 A_2 A_3 \, a_1 \, a_2 \, \mathfrak{A}_1 \\ b_1 \; b_2 \; b_3 \; B_1 B_2 \mathfrak{B}_1 \end{matrix}$$

$\begin{matrix}\mathfrak{A}_2 \\ \mathfrak{B}_2\end{matrix}$ is added to make $\sigma = 12$, and to this again $\begin{matrix}\mathfrak{A}_3 \\ \mathfrak{B}_3\end{matrix}$ in order to obtain $\sigma = 13$. The curve a^6, which is found when $\sigma = 13$, and the plane quartic $\bar{a}^4$, in α_{123}, when $\sigma = 11$, are both situated on the quartic surface α^4, which presents itself when $\sigma = 12$; therefore they meet, exclusive of $A_1 A_2 A_3$, in three other points. The surface α^9, associated with β, when $\sigma = 11$, is cut by a^6, (1) in the points A_1, A_2, A_3, quadruple on α^9; (2) in three points on $\bar{a}^4$, which is single on α^9, (3) in the six points associated with the intersections of b^6 and β, and (4) in eleven points lying on the twisted sextic curve, which is triple on α^9 and constitutes, with $\bar{a}^4$, the curve a_0^{10}. Hence the number of the intersections of a^6 with the entire curve a_0^{10} is $11 + 3 = 14 = 20 - 2k$, as above.

Similarly, in the other space $\mathbf{B}$, the curves $\bar{b}^4$, b_1, b_2, b_3, b_0^3 are cut by b^6 in $20 - 2l = 16$ points; which result agrees with the former, l being the number of the points B_i.

Of the above intersections, three lie in each of the lines $b_i (a_i)$ which form a constituent part of b_0^{10} (a_0^{10}).

46. We have now arrived at *the last case where* $\sigma = 14$. After having,

* When the signature is [331], these curves a^6, b^6 pass through the three points A^0, B^0 alluded to in No. 39.

if necessary, replaced a double condition by two single ones, we resolve (14) in $\left(11, \begin{smallmatrix}\mathfrak{A}_1\mathfrak{A}_2\\ \mathfrak{B}_1\mathfrak{B}_2\end{smallmatrix}\right)$ and $\left(11, \begin{smallmatrix}\mathfrak{A}_3\\ \mathfrak{B}_3\end{smallmatrix}\right)$. The latter gives in the space **A** a surface α^4, the former a curve a^6. This curve and surface intersect (1) in the k points A_i contained in (11), which are double on α^4, and single on a^6; (2) in $20-2k$ points, situated on a_0^{10}; and (3) in four other points, each of which has the same associated point for $\left(11, \begin{smallmatrix}\mathfrak{A}_1\mathfrak{A}_2\\ \mathfrak{B}_1\mathfrak{B}_2\end{smallmatrix}\right)$ as it has for $\left(11, \begin{smallmatrix}\mathfrak{A}_3\\ \mathfrak{B}_3\end{smallmatrix}\right)$, viz. the only point which is associated therewith for (11); each of these four points, therefore, has the point in question also associated with it for (14).

In the case of the signature

[340] $\begin{smallmatrix}A_1A_2A_3a_1a_2a_3a_4\\ b_1b_2b_3B_1B_2B_3B_4\end{smallmatrix}$

I substitute $\begin{smallmatrix}\mathfrak{A}_1\mathfrak{A}_1'\\ \mathfrak{B}_1\mathfrak{B}_1'\end{smallmatrix}$ in place of $\begin{smallmatrix}a_4\\ B_4\end{smallmatrix}$, and resolve into

(12) $\begin{smallmatrix}A_1A_2A_3a_1a_2a_4\\ b_1b_2b_3B_1B_2B_4'\end{smallmatrix}$, and (13) $\begin{smallmatrix}A_1A_2A_3a_1a_2a_3\mathfrak{A}_1\\ b_1b_2b_3B_1B_2B_3\mathfrak{B}_1'\end{smallmatrix}$,

which systems have in common

(11) $\begin{smallmatrix}A_1A_2A_3a_1a_2\mathfrak{A}_1\\ b_1b_2b_3B_1B_2\mathfrak{B}_1'\end{smallmatrix}$.

For (12) we have a cubic surface α^3; for (13) a sextic curve a^6. The curve $\bar{a}^4$, which we have to consider in (11), is not situated on α^3; therefore this surface is cut by a^6 (1) in the points A_i, single on α^3; (2) in eleven points situated on the sextic curve, which, with $\bar{a}^4$, constitutes the curve a_0^{10}; and (3) in four other points, each of which possesses an associated point for (14).

Hence we conclude that, *for all signatures where* $\sigma = 14$, *there are four points* $(A)_{14}$, *each of which possesses an associated point* $(B)_{14}$.*

Darmstadt, *March* 1876.

May 11*th*, 1876.

Prof. H. J. S. SMITH, F.R.S., President, in the Chair.

Dr. Lindemann (Editor of Dr. Clebsch's Correspondence) was present at the meeting.

The Rev. Dr. Logan was elected a member.

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* Dr. Hirst's researches have given an indirect confirmation of this unexpectedly simple result. A corresponding one is arrived at in my paper in the Math. Annalen, vol. x., p. 136.

Mr. Tucker communicated a paper by Mr. S. A. Renshaw, "On the Inscription of a Polygon in a Conic Section, subject to the condition that each of its sides shall pass through a given point by the aid of the generating circle of the conic." Prof. Cayley then spoke "On the Representation of Imaginary Quantities by an (n, n) Correspondence." Prof. Cayley having taken the chair, the President communicated two notes, one "On the Value of a Certain Determinant"; the other "On a Method of Solving the Pellian Equation."

The following presents were received :—

"Nuova Teoria delle soluzioni singolari delle equazioni differenziali di primo ordine e secondo grado tra due variabili," comunicazione del Prof. F. Casorati, Roma, 1876.

"Sulla Teoria delle soluzioni singolari delle equazioni differenziali"—Prof. F. Casorati.

"Zur Theorie der Algebraischen Flächen," von Rud. Sturm, in Darmstadt, Dec. 1875.

"Das Problem der Collineation" (R. Sturm, Math. Annalen, Dec. 1875).

"Bulletin des Sciences Mathématiques," Fév., Mars. 1876,—Tables des Matières et Noms d'Auteurs, Tome ix., 2^{me} sémestre, 1875.

"Note sur les singularités des courbes planes," par H. G. Zeuthen (Math. Ann.)

"The Physical Society's Proceedings," Part iii., Feb. 27 to June 26, 1875.

"Bulletin de la Société Mathématique de France," Tome iv., Mars, No. 3, 1876.

"Monatsbericht," Feb. 1876.

"Proceedings of Royal Society," Vol. xxiv., No. 169.

"Eine neue Relation zwischen den Singularitäten einer Algebraischen Curve," von Felix Klein, in München (Math. Ann., Jan. 1876).

"Educational Times," June 1876.

"Annali di Matematica," Serie ii^{a}, Tomo vii°, Fasc. 3° (Aprile 1876).

"Bulletin de l'Académie Royale des Sciences, des Lettres, et des Beaux-Arts de Belgique," 2^{e} série, Tomes 37, 38 (43^{me} année, 1874), Tomes 39, 40 (44^{me} année, 1875).

"Annuaire de l'Académie Royale des Sciences &c. de Belgique," 1875, 1876.

*Note on the Theory of the Pellian Equation, and of Binary Quadratic Forms of a Positive Determinant.** *By* Henry J. Stephen Smith, Savilian Professor of Geometry in the University of Oxford.

(*Read May 11th*, 1876.)

Art. 1. Let $\theta_0 = \mu_0 + \frac{1}{\mu_1} + \frac{1}{\mu_2} + \dots$ be any continued fraction, of which $\theta_1, \theta_2, \dots$ are the complete quotients; $\frac{p_0}{q_0}, \frac{p_1}{q_1}, \dots$ the successive convergents; so that

$$p_0 = \mu_0, \quad q_0 = 1, \quad \dots\dots \quad p_i = \mu_i p_{i-1} + p_{i-2}, \quad q_i = \mu_i q_{i-1} + q_{i-2},$$

$$\theta = \frac{\theta_i p_{i-1} + p_{i-2}}{\theta_i q_{i-1} + q_{i-2}}, \quad \theta_i = -\frac{p_{i-2} - \theta q_{i-2}}{p_{i-1} - \theta q_{i-1}}.$$

Also, let $p_{i-1} - \theta q_{i-1} = (-1)^i \epsilon_{i-1}$, so that $\theta_i = \frac{\epsilon_{i-2}}{\epsilon_{i-1}}$;

we have $\theta_1 = \frac{1}{\epsilon_0}$, and hence $\epsilon_{i-1} = \frac{1}{\theta_1 \theta_2 \dots \theta_i}$.

This expression, which supplies a measure of the rate of decrease of the difference ϵ_{i-1}, admits of an interesting application to the theory of the Pellian equation, and of binary quadratic forms of a positive determinant.

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* The following summary of the contents of this Note may be of use to the reader:—

Art. 1.—The relation, in a continued fraction, between the quantities ϵ and θ.

Art. 2.—The theorem that $T + U\sqrt{D}$ is equal to the product of the complete quotients in the development of $\sqrt{D}$.

Art. 3.—The same theorem for the period of complete quotients in the development of any quadratic surd.

Art. 4.—Theorems as to the number of different periods of complete quotients; viz., equations (1)—(4).

Art. 5.—Theorems as to the number of non-equivalent classes of quadratic forms; viz., equations (5) and (6).

Art. 6.—Equations arising from a comparison of the formulæ (5) and (6) with those of Dirichlet.

Arts. 7—13.—Discussion of the nature of the periods in the more important special cases.

Art. 14.—On the symmetry of any periodic series.

Art. 15.—On the arithmetical conditions under which the various special cases present themselves.

[It would be difficult to say that anything in the Addition (Arts. 7—15) is new: the discussion there attempted has never been given completely (see Art. 7); but this may have been because no one has thought it worth giving.]

2. Let D be any positive integer, not a perfect square. In the development of $\sqrt{D}$ in a continued fraction, let

$$\frac{\sqrt{D}+Q_1}{P_1}, \quad \frac{\sqrt{D}+Q_2}{P_2}, \quad \ldots\ldots \quad \frac{\sqrt{D}+Q_i}{P_i}$$

be the period of complete quotients; so that, if a is the integral number next inferior to $\sqrt{D}$,

$$\frac{\sqrt{D}+Q_1}{P_1} = \frac{\sqrt{D}+a}{D-a^2}, \quad \frac{\sqrt{D}+Q_i}{P_i} = \sqrt{D}+a.$$

Let T and U be the least integral numbers satisfying the equation

$$T^2 - DU^2 = (-1)^i,$$

and let $$\sqrt{D} = a + \frac{1}{\mu_1+}\,\frac{1}{\mu_2+}\cdots\frac{1}{\mu_{i-1}+}\,\frac{1}{2a+}\cdots;$$

we have $T = p_{i-1}$, $U = q_{i-1}$, $(-1)^i \epsilon_{i-1} = T - U\sqrt{D}$;

whence, by the preceding theorem,

$$T + U\sqrt{D} = \prod_{s=1}^{s=i} \frac{\sqrt{D}+Q_s}{P_s}.$$

Example.—The continued fraction equivalent to $\sqrt{13}$ is

$$\sqrt{13} = 3 + \frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{6+}\cdots\cdots,$$

and the period of complete quotients is

$$\frac{\sqrt{13}+3}{4}, \quad \frac{\sqrt{13}+1}{3}, \quad \frac{\sqrt{13}+2}{3}, \quad \frac{\sqrt{13}+1}{4}, \quad \sqrt{13}+3,$$

giving $$(\sqrt{13}+3)^2\,(\sqrt{13}+1)^2\,(\sqrt{13}+2) \times \frac{1}{144} = 18 + 5\sqrt{13},$$

and $$18^2 - 25 \times 13 = -1.$$

3. Again, let $$\Omega = a + 2bu + cu^2 = 0$$

represent any properly primitive equation of determinant D (*i.e.*, any quadratic equation whatever, in which a, b, c are integral numbers satisfying the equation $b^2 - ac = D$, and a, $2b$, c have no common divisor). If

$$\frac{\sqrt{D}+Q_1}{P_1}, \quad \frac{\sqrt{D}+Q_2}{P_2}, \quad \ldots\ldots \quad \frac{\sqrt{D}+Q_i}{P_i}$$

is the period of complete quotients obtained by the development of either root of Ω, we shall have, as before,

$$T + U\sqrt{D} = \prod_{s=1}^{s=i} \frac{\sqrt{D}+Q_s}{P_s}.$$

For the equations of the period are of the type,

$$(\Omega)\ldots\ldots\begin{cases} a_0+2\beta_0u_0-a_1u_0^2=0, \\ -a_1-2\beta_1u_1+a_2u_1^2=0, \\ a_2+2\beta_2u_2-a_3u_2^2=0, \\ \ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots \\ (-1)^{i-1}a_{i-1}+(-1)^{i-1}\,2\beta_{i-1}\,u_{i-1}+(-1)^i\,a_0u_{i-1}^2=0, \\ (-1)^i[a_0+2\beta_0u_0-a_1u_0^2\,]=0, \\ \ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots \end{cases}$$

so that, if $\mu_0, \mu_1, \mu_2, \ldots\ldots \mu_{i-1}$, is the period of integral quotients, we have

$$u_0=\mu_0+\frac{1}{\mu_1+}\ldots\ldots\frac{1}{\mu_{i-1}+}\frac{1}{u_0},$$

where

$$u_0=\frac{\beta_0+\sqrt{D}}{a_1}.$$

Hence

$$\frac{1}{\epsilon_{i-1}}=\frac{(-1)^i}{p_{i-1}-q_{i-1}u_0},$$
$$=u_0u_1u_2\ldots u_{i-1},$$
$$=\prod_{s=1}^{s=i}\frac{\sqrt{D}+Q_s}{P_s}.$$

But, by a known theorem (see the Report on the Theory of Numbers, in the Report of the British Association for 1861, Art. 96 (i.), p. 315), we have

$$(Q.)\ldots\ldots\ldots\ldots\begin{cases} T=\frac{1}{2}(p_{i-1}+q_{i-2}), \\ U=\frac{p^{i-2}}{a_0}=\frac{p_{i-1}-q_{i-2}}{2\beta_0}=\frac{q_{i-1}}{a_1}; \end{cases}$$

whence

$$p_{i-1}-q_{i-1}u_0=p_{i-1}-q_{i-1}\frac{\beta_0+\sqrt{D}}{a_1}=T-U\sqrt{D},$$

or

$$T+U\sqrt{D}=\frac{(-1)^i}{T-U\sqrt{D}}=\prod_{s=i}^{s=i}\frac{\sqrt{D}+Q_s}{P_s}.$$

If the given equation (Ω) is improperly primitive (*i.e.*, if the numbers a, $2b$, c have 2 for their greatest common divisor), we have to replace the numbers T and U in the equations (Q), by $\frac{1}{2}T_1$ and $\frac{1}{2}U_1$, where T_1 and U_1 are the least numbers which satisfy the equation $T_1^2-DU_1^2=(-1)^i4$; and we find

$$\tfrac{1}{2}(T_1+U_1\sqrt{D})=\prod_{s=1}^{s=i}\frac{\sqrt{D}+Q_s}{P_s}.$$

4. Every primitive quadratic equation of determinant D, of which one root is positive and greater than unity, and the other negative and less in absolute magnitude than unity, occurs in one, and only in one, of the periods of equations of determinant D (see the Report cited, p. 309, Art. 93). Hence, every expression of the form $\frac{\sqrt{D}+Q}{P}$, in which, P and Q being positive, Q is less than $\sqrt{D}$, P is a divisor of $D-Q^2$ intermediate between $\sqrt{D}-Q$ and $\sqrt{D}+Q$, and the three numbers

P, Q, $\frac{D-Q^2}{P}$ are relatively prime, is the root of an equation contained in a period of equations of determinant D; and, for any given determinant, the number of periods of complete quotients is equal to the number of periods of quadratic equations, if we regard two quadratic equations such as

$$a+2b\theta+c\theta^2=0,$$
$$-a-2b\theta-c\theta^2=0,$$

which differ only in sign, as identical with one another.

Let $P'=\frac{D-Q^2}{P}$, and let k be the number of periods of properly primitive complete quotients of determinant D; we have evidently

$$(1)\ldots\ldots\ldots\ldots\ldots\ldots(T+U\sqrt{D})^k=\Pi\frac{\sqrt{D}+Q}{P},$$

or
$$k=\frac{1}{\log(T+U\sqrt{D})}\Sigma\log\frac{\sqrt{D}+Q}{P},$$

the sign of multiplication Π, and the sign of summation Σ, extending to all positive numbers Q which do not surpass $\sqrt{D}$, and to all divisors P of $D-Q^2$, which are intermediate between $\sqrt{D}+Q$ and $\sqrt{D}-Q$, and are such that the three numbers P, 2Q, P′ admit of no common divisor other than unity. Let $\psi(Q)$ be the number of such divisors of $D-Q^2$; observing that, if $\frac{\sqrt{D}+Q}{P}$ is a complete quotient in a period, $\frac{\sqrt{D}+Q}{P'}=\frac{P}{\sqrt{D}-Q}$ is also a complete quotient in the same or in a different period, we may write

$$(2)\ldots\ldots\ldots\ldots(T+U\sqrt{D})^k=\Pi\left[\frac{\sqrt{D}+Q}{\sqrt{D}-Q}\right]^{\frac{1}{2}\psi(Q)};$$

where it will be noticed that, if $P=P'$, $D=P^2+Q^2$, the two identical complete quotients $\frac{\sqrt{D}+Q}{P}$ and $\frac{P}{\sqrt{D}-Q}$ are each of them equal to $\left[\frac{\sqrt{D}+Q}{\sqrt{D}-Q}\right]^{\frac{1}{2}}$.

When $D\equiv 1$, mod. 4, let k_1 be the number of periods of improperly primitive complete quotients of determinant D, we find as before

$$(3)\ldots\ldots\ldots\ldots\ldots(\tfrac{1}{2}T_1+\tfrac{1}{2}U_1\sqrt{D})^{k_1}=\Pi\frac{\sqrt{D}+Q}{P},$$

or
$$k_1=\frac{1}{\log\frac{1}{2}(T_1+U_1\sqrt{D})}\Sigma\log\frac{\sqrt{D}+Q}{P},$$

the symbols Π and Σ extending to all positive uneven numbers Q which are less than $\sqrt{D}$, and to all divisors P of $D-Q^2$ which are intermediate between $\sqrt{D}+Q$ and $\sqrt{D}-Q$, and are such that the three numbers P, 2Q, P′ have 2 for their greatest common divisor. If

$\psi_1(Q)$ be the number of such divisors of $D-Q^2$, we may also write

$$(4)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\tfrac{1}{2}T_1+\tfrac{1}{2}U_1\sqrt{D})^{h_1} = \Pi\left[\frac{\sqrt{D}+Q}{\sqrt{D}-Q}\right]^{\frac{1}{2}\psi_1(Q)}.$$

It will be observed that, if Q is even, we have always $\psi_1(Q) = 0$.

5. Let h and h_1 respectively denote the numbers of properly and improperly primitive classes of quadratic forms of determinant D, and let $[r, v]$, $[r_1, v_1]$ be the least numbers which satisfy the equations $r^2-Dv^2 = +1$, $r_1^2-Dv_1^2 = +4$; so that, when the equations

$$(T)\ldots\ldots\ldots\ldots\ldots\ldots\ldots x^2-Dy^2 = -1,\quad x^2-Dy^2 = -4,$$

are resoluble [they are either both resoluble or both irresoluble],

$$r+v\sqrt{D} = (T+U\sqrt{D})^2,\quad \tfrac{1}{2}r_1+\tfrac{1}{2}v_1\sqrt{D} = (\tfrac{1}{2}T_1+\tfrac{1}{2}U_1\sqrt{D})^2;$$

and when the equations (T) are irresoluble,

$$r+v\sqrt{D} = T+U\sqrt{D},\quad r_1+v_1\sqrt{D} = T_1+U_1\sqrt{D}.$$

We can now establish the equations

$$(5)\ldots\ldots\ldots\ldots\ldots\ldots\ldots (r+v\sqrt{D})^h = \Pi\left[\frac{\sqrt{D}+Q}{\sqrt{D}-Q}\right]^{\psi(Q)},$$

$$(6)\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\tfrac{1}{2}r_1+\tfrac{1}{2}v_1\sqrt{D})^{h_1} = \Pi\left[\frac{\sqrt{D}+Q}{\sqrt{D}-Q}\right]^{\psi_1(Q)}.$$

For this purpose it is only necessary to show that, when the equations (T) are not resoluble, we have,

$$(a)\ldots\ldots\ldots\ldots\ldots\ldots h = 2k,\quad h_1 = 2k_1;$$

but when these equations are resoluble, we have, instead,

$$(b)\ldots\ldots\ldots\ldots\ldots\ldots h = k,\quad h_1 = k_1.$$

To the single period of complete quotients

$$(7)\ldots\ldots\ldots\ldots\ldots \frac{\sqrt{D}+\beta_0}{\alpha_1},\quad \frac{\sqrt{D}+\beta_1}{\alpha_2},\quad \frac{\sqrt{D}+\beta_2}{\alpha_3},\ldots\ldots$$

there correspond two periods of reduced quadratic forms, viz.,

$$(8)\ldots\ldots\ldots\ldots\ldots (\alpha_0, \beta_0, -\alpha_1),\quad (-\alpha_1, \beta_1, \alpha_2),\quad (\alpha_2, \beta_2, -\alpha_3),$$
$$\ldots\qquad\ldots\qquad\ldots\qquad\ldots\qquad\ldots$$

and

$$(9)\ldots\ldots\ldots\ldots\ldots (-\alpha_0, \beta_0, \alpha_1),\quad (\alpha_1, \beta_1, -\alpha_2),\quad (-\alpha_2, \beta_2, \alpha_3),$$
$$\ldots\qquad\ldots\qquad\ldots\qquad\ldots\qquad\ldots$$

For the complete quotients (7) are the positive roots of the equations (Ω); they are also the positive roots of the same equations with their signs changed; and the period (9) is related to the period $(-\Omega)$ exactly as the period (8) is related to the period (Ω); viz., the coefficients of the forms are the same as the coefficients of the equations, except that in the periods of equations the middle coefficients are alternately positive and negative, whereas in the periods of forms these coefficients are all positive. The two periods of forms are, in general, but

not always, distinct; and we shall now prove (what is indeed well known) that these two periods are, or are not, identical, according as the equations (T) are, or are not, resoluble. We may observe that the form $(a, -b, c)$ is termed the *opposite* of the form (a, b, c), and $(-a, -b, -c)$ the *negative* of (a, b, c); thus $(-a, b, -c)$ is the negative of the opposite of (a, b, c).

(i.) If the equations (T) are resoluble, any form (a, b, c) of determinant D is properly equivalent to the negative of its opposite; viz., (a, b, c) is transformed into $(-a, b, -c)$ by

$$\begin{vmatrix} b\mathrm{U}-\mathrm{T}, & -c\mathrm{U} \\ -a\mathrm{U}, & b\mathrm{U}+\mathrm{T} \end{vmatrix}.$$

Hence the reduced forms $(\alpha_0, \beta_0, -\alpha_1)$, $(-\alpha_0, \beta_0, \alpha_1)$ are properly equivalent; either of them is therefore contained in the period of the other; *i.e.*, the two periods are identical.

(ii.) If the two periods (8) and (9) are identical, the form $(-\alpha_0, \beta_0, \alpha_1)$ must occur in the period of $(\alpha_0, \beta_0, -\alpha_1)$; and because its first coefficient is negative, it must occupy an even place in that period. Hence the period of complete quotients (7) consists of an uneven number of terms; and we infer from the formulæ (Q) of Art. 3 that the equations (T) are resoluble.

6. If $\mathrm{D} \equiv 1$, mod. 8, we have $h = h_1$; if $\mathrm{D} \equiv 5$, we have $h = 3h_1$ when τ_1 and v_1 are even, but $h = h_1$ when τ_1 and v_1 are uneven, in which case

$$(\tfrac{1}{2}\tau_1 + v_1\sqrt{\mathrm{D}})^3 = \tau + v\sqrt{\mathrm{D}}.$$

We thus find, in both cases alike,

$$(12) \quad \ldots\ldots\ldots\ldots \quad \Pi\left[\frac{\sqrt{\mathrm{D}}+\mathrm{Q}}{\sqrt{\mathrm{D}}-\mathrm{Q}}\right]^{\psi(\mathrm{Q})} = \Pi\left[\frac{\sqrt{\mathrm{D}}+\mathrm{Q}}{\sqrt{\mathrm{D}}-\mathrm{Q}}\right]^{\sigma\psi_1(\mathrm{Q})},$$

where $\sigma = 1$, or $= 3$, according as $\mathrm{D} \equiv 1$, or $\equiv 5$, mod. 8.

Again, since, by the formulæ of Lejeune Dirichlet, we have (see the Report cited, Art. 101)

$$h \log [\tau + v\sqrt{\mathrm{D}}] = 2\sqrt{\mathrm{D}}\ \Sigma\left(\frac{\mathrm{D}}{n}\right)\frac{1}{n},$$

where the sign of summation extends to all numbers prime to 2D, and $\left(\frac{\mathrm{D}}{n}\right)$ is the generalized symbol of quadratic reciprocity, we obtain

$$2\sqrt{\mathrm{D}}\ \Sigma\left(\frac{\mathrm{D}}{n}\right)\frac{1}{n} = \Sigma\psi(\mathrm{Q}) \log\frac{\sqrt{\mathrm{D}}+\mathrm{Q}}{\sqrt{\mathrm{D}}-\mathrm{Q}},$$

or

$$(13) \quad \ldots\ldots\ldots\ldots \Sigma\left(\frac{\mathrm{D}}{n}\right)\frac{1}{n} = \Sigma\psi(\mathrm{Q})\left[\frac{\mathrm{Q}}{\mathrm{D}} + \frac{\mathrm{Q}^3}{3\mathrm{D}^2} + \frac{\mathrm{Q}^5}{5\mathrm{D}^3} + \ldots\ldots\right].$$

Similarly, from the formulæ (see *ibid.*)

$$h_1 \log [\tfrac{1}{2}\tau_1 + \tfrac{1}{2}v_1\sqrt{\mathrm{D}}] = \frac{2}{\sigma}\sqrt{\mathrm{D}}\ \Sigma\left(\frac{\mathrm{D}}{n}\right)\frac{1}{n},$$

where σ has the same meaning as before, we infer

$$(14)\ldots\ldots\Sigma\left(\frac{D}{n}\right)\frac{1}{n}=\sigma\Sigma\psi_1(Q)\left[\frac{Q}{D}+\frac{Q^3}{3D^2}+\frac{Q^5}{5D^3}+\ldots\ldots\right].$$

It is probable that a direct demonstration of the equations (12), (13), (14), of which any two involve the third, would offer considerable difficulties.

Addition to the preceding Note.

7. As the preceding determination (equations 5 and 6) of the number of non-equivalent classes for a positive determinant depends on the equations (a) and (b), which assign the relation between the number of periods of complete quotients and the number of periods of reduced forms, it is worth while, for the sake of distinctness, to describe fully the characteristic appearances presented by these periods in certain special cases which are of some importance.

Every form, or class of forms, is, of course, properly equivalent to itself, and improperly equivalent to its opposite. But a form, or class of forms, may be—

(i.) Properly equivalent to its opposite, and improperly equivalent to itself (in this case the class is ambiguous);

(ii.) Properly equivalent to its negative, and improperly equivalent to the negative of its opposite;

(iii.) Properly equivalent to the negative of its opposite, and improperly equivalent to its negative.

Since, if any two of these specialities coexist, they necessarily involve the third, there are four cases to be considered, viz., the cases (i.), (ii.), (iii.), in which the specialities (i.), (ii.), (iii.) exist singly, and the case (iv.) in which they all exist simultaneously. We shall briefly refer to each of these cases in succession. We may observe, however, that the case (i.), which is that of an ambiguous class, has been fully considered by Gauss (*Disq. Arith.*, Art. 187, Obs. 6, 7, 8); of the rest, the case (ii.) has, perhaps, attracted less attention than might have been expected.

8. If the period of reduced forms equivalent to (a, b, c) is

$$(8)\ \ldots\ldots\ (a_0, \beta_0, -a_1),\ (-a_1, \beta_1, a_2)\ \ldots\ldots\ (-a_{2k-1}, \beta_{2k-1}, a_0),$$

the periods of reduced forms equivalent to $(a, -b, c)$, $(-a, -b, -c)$, $(-a, b, -c)$ are respectively

$$(10)\ldots\ldots(a_0, \beta_{2k-1}, -a_{2k-1}),\ (-a_{2k-1}, \beta_{2k-2}, a_{2k-2})\ \ldots\ldots\ (-a_1, \beta_0, a_0),$$

$$(11)\ldots\ldots(-a_0, \beta_{2k-1}, a_{2k-1}),\ (a_{2k-1}, \beta_{2k-2}, -a_{2k-2})\ \ldots\ldots\ (a_1, \beta_0, -a_0),$$

$$(9)\ldots\ldots(-a_0, \beta_0, a_1),\qquad (a_1, \beta_1, -a_2)\ \ldots\ldots\ldots\ (a_{2k-1}, \beta_{2k-1}, -a_0).$$

As in Art. 3, we designate the period (7) of complete quotients, or, which is the same thing, the period formed by the positive roots of the equations (Ω), by $u_0, u_1, \ldots\ldots u_{2k-1}$.

The negative roots of the same equations we represent by

$$-\frac{1}{v_0}, \quad -\frac{1}{v_1}, \quad \ldots\ldots \quad -\frac{1}{v_{2k-1}};$$

so that, if $\mu_0, \mu_1, \ldots\ldots \mu_{2k-1}$ is the period of integral quotients, we have, by a well-known theorem,

$$\mu_s = \mathrm{I}u_s = \mathrm{I}v_{s+1},$$

the symbol $\mathrm{I}x$ denoting the greatest integral number not surpassing x. The four periods of forms (8), (10), (11), (9), we represent for brevity by the symbols

$$\begin{array}{lllll} \Phi & \ldots\ldots \phi_0, & \phi_1, & \phi_2, & \ldots\ldots \phi_{2k-1}, \\ \Psi & \ldots\ldots \psi_{2k-1}, & \psi_{2k-2}, & \psi_{2k-3}, & \ldots\ldots \psi_0, \\ \overline{\Psi} & \ldots\ldots \overline{\psi}_{2k-1}, & \overline{\psi}_{2k-2}, & \overline{\psi}_{2k-3}, & \ldots\ldots \overline{\psi}_0, \\ \overline{\Phi} & \ldots\ldots \overline{\phi}_0, & \overline{\phi}_1, & \overline{\phi}_2, & \ldots\ldots \overline{\phi}_{2k-1}. \end{array}$$

Two such forms as (a, b, c), (c, b, a) are said to be *associated;* thus, ϕ_s and ψ_s, or again $\overline{\phi}_s$ and $\overline{\psi}_s$, are associated forms. The periods Φ and Ψ, and again the periods $\overline{\Phi}$ and $\overline{\Psi}$, are themselves termed associated periods.

The period of complete quotients corresponding to the periods Φ and $\overline{\Phi}$ is (Art. 5) $\quad u_0, u_1, \ldots\ldots u_{2k-1};$

and similarly the period of complete quotients corresponding to the periods Ψ and $\overline{\Psi}$ is $\quad v_{2k-1}, v_{2k-2}, \ldots\ldots v_1, v_0.$

9. Case (i.)—If (a, b, c) is properly equivalent to $(a, -b, c)$, the periods Φ and Ψ must coincide; *i.e.*, we must have, for some value of σ, $\phi_0 = \psi_{2\sigma+1}$, the suffix $2\sigma+1$ being uneven, because the extreme coefficients of $\psi_{2\sigma+1}$ must have the same signs as the extreme coefficients of ϕ_0. Hence $\phi_1 = \psi_{2\sigma}$, $\phi_2 = \psi_{2\sigma-1}$, ……; and finally $\phi_\sigma = \psi_{\sigma+1}$, $\phi_{\sigma+1} = \psi_\sigma$, or there occur in the period two consecutive forms of which each is the associate of the other. As we may begin the period with any form we please, we may suppose that ϕ_0 and ϕ_1 are these two consecutive forms, so that $\beta_1 = \beta_0$, $a_2 = a_0$, $2\beta_0 \equiv 0$, mod. a_1. It will be seen that, if $\phi_0 = \psi_{2\sigma+1}$, we have not only $\phi_\sigma = \psi_{\sigma+1}$, $\phi_{\sigma+1} = \psi_\sigma$, but also

$$\phi_{\sigma+k} = \psi_{\sigma+1-k} = \psi_{\sigma+k+1}, \quad \phi_{\sigma+k+1} = \psi_{\sigma+k}.$$

Thus a sequence of two associated forms occurs twice in the period; and, assuming (as we have done) that $\sigma=0$, the period of forms is of the type $\quad \psi_1, \phi_1;\ \phi_2, \phi_3 \ldots \phi_{k-1};\ \phi_k, \psi_k;\ \psi_{k-1}, \psi_{k-2}, \ldots \psi_2,$

where $\phi_{s+1} = \psi_{2k-s}$ for every value of s. The period of integral quotients is of the type

$$\lambda_0, \mu_1, \mu_2, \ldots \mu_{k-1};\ \lambda_k, \mu_{k-1}, \mu_{k-2}, \ldots \mu_1,$$

where $$\lambda_0 = \frac{2\beta_0}{a_1}, \quad \lambda_k = \frac{2\beta_k}{a_{k+1}}.$$

The period of complete quotients is of the type

$$v_1, u_1 ; u_2, u_3, \ldots u_{k-1} ; u_k, v_k ; v_{k-1}, v_{k-2}, \ldots v_2,$$

where $u_{s+1} = v_{2k-s}$ for every value of s. The periods of the coefficients α and β are respectively of the types

$$\alpha_1 ; \alpha_2 \ldots \alpha_k ; \alpha_{k+1} ; \alpha_k \ldots \alpha_2 ;$$
$$\beta_1, \beta_1 ; \beta_2, \ldots \beta_{k-1} ; \beta_k, \beta_k ; \beta_{k-1}, \ldots \beta_2.$$

The two *ambiguous* forms are ϕ_1 and ψ_k.

10. Case (ii.)—If (a, b, c) is properly equivalent to $(-a, -b, -c)$, the periods Φ and $\overline{\Psi}$ must coincide. Hence we must have (for some even value of the suffix 2σ) $\phi_0 = \overline{\psi}_{2\sigma}$; whence $\phi_\sigma = \overline{\psi}_\sigma$, and also $\phi_{\sigma+k} = \overline{\psi}_{\sigma-k} = \overline{\psi}_{\sigma+k}$. The equation $\phi_\sigma = \overline{\psi}_\sigma$ is equivalent to the equation $\alpha_\sigma = \alpha_{\sigma+1}$; we thus see that the period contains two forms, in each of which the extreme coefficients are equal in absolute magnitude; so that (supposing, as before, that $\sigma = 0$), we have

$$D = \beta_0^2 + \alpha_0^2, \quad D = \beta_k^2 + \alpha_k^2.$$

The period of reduced forms is of the type

$$\overline{\psi}_0 = \phi_0 ; \phi_1, \phi_2, \ldots \phi_{k-1} ; \phi_k = \overline{\psi}_k ; \overline{\psi}_{k-1}, \overline{\psi}_{k-2} \ldots \overline{\psi}_1,$$

where $\varphi_s = \overline{\psi}_{2k-s}$; the period of complete quotients is of the type

$$u_0 = v_0 ; u_1, u_2, \ldots u_{k-1} ; u_k = v_k ; v_{k-1}, v_{k-2}, \ldots v_1,$$

where $u_s = v_{2k-s}$.

Lastly, the periods of integral quotients and of the coefficients α and β are of the types

$$\mu_0, \mu_1, \ldots \mu_{k-1} ; \mu_{k-1}, \mu_{k-2}, \ldots \mu_1, \mu_0 ;$$
$$\alpha_1, \alpha_2 \ldots \alpha_{k-1}, \alpha_k, \alpha_k, \alpha_{k-1}, \ldots \alpha_2, \alpha_1 ;$$
$$\beta_0 ; \beta_1, \beta_2 \ldots \beta_{k-1} ; \beta_k ; \beta_{k-1}, \beta_{k-2} \ldots \beta_1.$$

11. Case (iii.)—If (a, b, c) is properly equivalent to $(-a, b, -c)$, the periods Φ and $\overline{\Phi}$ coincide, and we must have $\phi_0 = \overline{\phi}_{2\sigma+1}$, $\overline{\phi}_0 = \phi_{2\sigma+1}$, where $2\sigma+1$ is less than $2k$. From these equations we infer $\overline{\phi}_0 = \overline{\phi}_{2(2\sigma+1)}$, or $2\sigma+1 = k$, since if $\overline{\phi}_0 = \phi_m$, m is a multiple of $2k$. The period of forms is therefore of the type

$$\phi_0, \phi_1, \ldots \phi_{k-1}, \overline{\phi}_0, \overline{\phi}_1, \ldots \overline{\phi}_{k-1},$$

k being an uneven number; the periods of complete quotients, of integral quotients, and of the coeffic ents α and β, consist each of a period of k terms, twice repeated. The period of equations (Ω) in like manner consists of a period of k equations, twice repeated; but each equation appears in the second half of the period with its sign changed.

12. Case (iv.)—If (a, b, c) is properly equivalent to any two of the forms $(a, -b, c)$, $(-a, -b, -c)$, $(-a, b, c)$, and therefore to all three of them, the nature of the periods is most readily ascertained by considering the series of integral quotients. Since the conditions characteristic of the cases (i.) and (iii.) must be united, the semi-period

$$\lambda_0, \mu_1, \mu_2, \ldots\ldots \mu_{k-1}$$

must be term for term identical with the semi-period

$$\lambda_k, \mu_{k-1}, \mu_2, \ldots\ldots \mu_1,$$

k being an uneven number $2i+1$. Hence $\lambda_0 = \lambda_k$, and the period is of the type

$$\lambda, \mu_1, \mu_2 \ldots \mu_i, \mu_i \ldots \mu_2, \mu_1,$$

twice repeated; which combines the characters of the periods of integral quotients in the cases (i.), (ii.), (iii.)

The period of forms is of the type

$$\psi_1, \phi_1; \phi_2, \phi_3, \ldots \phi_{i+1}, \overline{\psi}_i, \overline{\psi}_{i-1}, \ldots \overline{\psi}_2;$$
$$\overline{\psi}_1, \overline{\phi}_1; \overline{\phi}_2, \overline{\phi}_3, \ldots \overline{\phi}_{i+1}, \psi_i, \psi_{i-1}, \ldots \psi_2;$$

where

$$\phi_s = \overline{\psi}_{2i+2-s} = \overline{\phi}_{s-2i-1} = \psi_{4i+3-s};$$

the period of complete quotients is of the type

$$v_1, u_1; u_2, u_3, \ldots u_{i+1}; v_i, v_{i+1}, \ldots v_2;$$

twice repeated, where $u_s = v_{2\ i2-s}$; and the periods of the coefficients α and β are respectively

$$\alpha_1; \alpha_2, \alpha_3, \ldots \alpha_{i+1}, \alpha_{i+1}, \alpha_i, \ldots \alpha_2;$$
$$\beta_1, \beta_1; \beta_2, \ldots \beta_i, \beta_{i+1}, \beta_i, \ldots \beta_2;$$

each of them twice repeated.

13. If therefore we develope the two roots of a given primitive quadratic equation, we obtain, in the general case, two distinct associated periods of complete quotients and four distinct periods of reduced forms. In the special cases (i.) and (ii.), we have but one period of complete quotients and two periods of reduced forms; in case (i.) the two associated periods of each pair combine; in case (ii.) each period of reduced forms becomes identical with the negative of the opposite of its associated period. In case (iii.) we have two distinct periods of complete quotients; but only two distinct periods of reduced forms; the period of any form being identical with the period of the opposite of its negative, and consisting of an uneven number of forms followed by the opposites of the negatives of the same forms; the period of complete quotients contains only half as many terms as the period of reduced forms. Lastly, in case (iv.) we have but one period of complete quotients, and but one period of reduced forms, the four periods, which in the general case are distinct, being all identical with one another.

We may observe that if the equation

$$(p) \ldots\ldots\ldots\ldots \nu a - 2\mu b + \lambda c = 0$$

can be satisfied by three numbers λ, μ, ν, which also satisfy the condition

$$(q) \ldots\ldots\ldots\ldots \mu^2 - \lambda\nu = 1,$$

the form (a, b, c) is transformed into $(a, -b, c)$ by the substitution $\begin{vmatrix} \mu, & -\nu \\ -\lambda, & \mu \end{vmatrix}$, and consequently has a period of the type (i.) If, instead of the condition (q), the condition

$$(r) \ldots\ldots\ldots\ldots \mu^2 - \lambda\nu = -1,$$

is satisfied by the three numbers (λ, μ, ν), (a, b, c) is transformed into $(-a, -b, -c)$ by $\begin{vmatrix} \mu, & \nu \\ -\lambda, & -\mu \end{vmatrix}$ and the period of (a, b, c) is of the type (ii.) Lastly, if the equation (p) can be satisfied by two different sets of numbers, of which one set satisfies the condition (q) and the other the condition (r), the period of (a, b, c) is of the type (iv.)

14. The periods which we have to consider in the cases (i.), (ii.) and (iv.) afford examples of each of the three kinds of symmetry which can exist in a periodic series. Let $\ldots c_0 c_1 c_2 \ldots c_{n-1} \ldots$ be a period of n terms repeated indefinitely in both directions; it will be found that the series thus formed may be symmetrical (*i.e.*, may be the same whether we follow it forwards or backwards) in three and only in three different ways.

(i.) Let n be even; and let the series continued from c_0 forwards coincide with the series continued from c_{2k+1} backwards so that $c_0 = c_{2k+1}$, $c_1 = c_{2k}$, ... : the period then is

$$c_0 c_1, \ldots c_k; \ c_k, \ldots c_0, \ c_{2k+2}, \ldots c_{n-1};$$

or, if $n = 2\nu$,

$$(A) \ldots\ldots\ldots c_{-\nu+k+1}, c_{-\nu+k+2}, \ldots c_0, \ c_1, \ldots c_k, \ c_k, \ldots c_0, \ c_{-1}, \ldots, c_{-\nu+k+1};$$

where there are two centres of symmetry, one falling between the two terms c_k, the other falling between the two terms $c_{-\nu+k+1}$.

Of this type is the period of the coefficients β in case (i.); and the periods of the integral quotients and of the coefficients α in case (ii.)

(ii.) Let n still be even, but let the series continued forwards from c_0 coincide with the series continued backwards from c_{2k}; the period is

$$c_0, \ c_1, \ \ldots \ c_{k-1}, \ \ c_k, \ c_{k-1}, \ \ldots \ c_0, \ \ c_{2k+1}, \ \ldots \ c_{n-1},$$

or, if $n = 2\nu$,

$$(B) \ldots\ldots c_{-\nu+k}; \ c_{-\nu+k+1} \ldots\ldots \ c_0, \ c_1 \ldots\ldots \ c_{k-1}; \ c_k; \ c_{k-1} \ldots\ldots \ c_0, \ \ldots\ldots \ c_{-\nu+k+1},$$

where there are two centres of symmetry falling on the terms $c_{-\nu+k}$ and c_k respectively. The symmetry of the integral quotients, and of the

coefficients α in case (i.), and of the coefficients β in case (ii.) is of this type.

(iii.) Let $n = 2\nu+1$ be uneven; and let the series continued forward from c_0 coincide with the series continued backward from c_{2k+1}. The period is of the type

$$c_0, c_1, \ldots c_k, \quad c_k \ldots c_0, \quad c_{2k+2}, \ldots c_{n-1},$$

or $$c_{-\nu+k}; c_{-\nu+k+1}, c_{-\nu+k+2}, \ldots c_0, c_1 \ldots c_k, c_k \ldots c_0, c_{-1} \ldots c_{-\nu+k+1},$$

where again there are two centres of symmetry, one falling on the term $c_{-\nu+k}$, the other between the two terms c_k. If we had supposed $c_0 = c_{2k}$ we should have obtained a period of the same form. It is evident that, if this period be doubled, it combines the symmetries of the periods (A) and (B). Of this type are the periods of integral quotients and of the coefficients α and β in case (iv.) In case (iii.) there is no symmetry; but an unsymmetrical uneven period is twice repeated.

15. Every determinant has ambiguous classes; and every ambiguous class has a period of the type (i.) But developments of the types (ii.), (iii.), (iv.) can only present themselves in the case of determinants of the form P or 2P, where P is a product of uneven prime numbers of the form $4n+1$. For in case (ii.) D must, as we have seen, be the sum of two square numbers prime to one another, and in case (iii.) the equation $T^2 - DU^2 = -1$ must be resoluble, whence again D is the sum of two squares prime to one another.

If μ is the number of different primes dividing P, the number of ways in which P can be decomposed into the sum of an even and uneven square prime to one another is $2^{\mu-1}$. Let $D = P$, and let $D = A^2 + B^2$ be one of these decompositions, A being uneven and B even; the forms $(-A, B, A)$, $(A, B, -A)$, $(-B, A, B)$, $(B, +A, -B)$ are all reduced, and the first two are properly, the last two improperly, primitive. We thus have $2^{\mu-1}$ properly primitive periods of reduced forms of the type (ii.); and as many improperly primitive periods of the same type; *i.e.*, since there are $2^{\mu-1}$ properly and as many improperly primitive ambiguous classes, there are as many classes having periods of the type (ii.) as there are classes having periods of the type (i.)

If $D = 2P$, we have a similar result; viz., there are $2^{\mu-1}$ equations of the form $D = 2P = A^2 + B^2$, in which A and B are both uneven. We thus obtain 2^{μ} properly primitive periods of reduced forms of the type (ii.); *i.e.*, as many as there are of type (i.) There are, of course, no improperly primitive classes of a determinant of the form 2P.

When the equations (T) are not resoluble, but the determinant is of either of the forms P or 2P, the developments of the type (i.) and those of the type (ii.) are entirely distinct from one another. On the other hand, when the equations (T) are resoluble, the developments of

the types (i.) and (ii.) coincide, giving rise to developments of the type (iv.), and all the remaining developments are of the type (iii.)

It is known that, when D is an uneven power of an uneven prime of the form $4n+1$, the equations (T) are always resoluble. But when D has any other value of either of the forms P or 2P, there is no known criterion for deciding whether these equations are or are not resoluble.

On the Value of a Certain Arithmetical Determinant. By Henry J. Stephen Smith, Savilian Professor of Geometry in the University of Oxford.

[*Read May 11th*, 1876.]

Let (m, n) denote the greatest common divisor of the integral numbers m and n; and let $\psi(m)$ be the number of numbers not surpassing m and prime to m; the symmetrical determinant

$$\Delta_m = \Sigma \pm (1,1)(2,2) \ldots\ldots (m,m)$$

is equal to $$\psi(1) \times \psi(2) \times \ldots\ldots \times \psi(m).$$

This theorem may be established as follows. Let $p_1, p_2, \ldots$ be all the different primes dividing m, and consider the columns (P) of which the indices are

$$m, \quad \frac{m}{p_1}, \quad \frac{m}{p_2}, \ldots, \quad \frac{m}{p_1 p_2}, \ldots, \quad \frac{m}{p_1 p_2 p_3}, \ldots\ldots$$

Take these columns with the signs of the corresponding terms in the product $$\psi(m) = m\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\ldots\ldots;$$

and, attending to these signs, replace the terms of the last column of Δ_m by the sum of the corresponding terms in the columns (P). The value of Δ_m is not changed: the term (m, m) is evidently replaced by $\psi(m)$; and we shall now show that every other term (m, k) in the last column is replaced by zero; *i.e.*, that $\Delta_m = \psi(m) \times \Delta_{m-1}$, which is the theorem to be proved.

First, let k be prime to m; then

$$(m, k) = 1, \quad \left(\frac{m}{p}, \ k\right) = 1, \quad \&c.;$$

and (m, k) has to be replaced by a sum of units, of which as many are negative as positive; *i.e.*, by zero.

Secondly, let k be a divisor of m, other than unity or m itself; and let us separate the primes p into two classes, q and r, in such a manner that k does not divide any quotient of the form $\frac{m}{q}$, but does divide every quotient of the form $\frac{m}{r}$. There may or may not be any primes q, but there must be at least one prime r, or we should have $k = m$: we further observe that

$$(m, k) = \left(\frac{m}{r}, k\right) = \left(\frac{m}{r_1 r_2}, k\right) = \ldots\ldots = k;$$

$$\left(\frac{m}{q}, k\right) = \frac{k}{q}, \left(\frac{m}{q_1 q_2}, k\right) = \frac{k}{q_1 q_2}, \ldots\ldots$$

Thus, if we were to attend only to those columns of which the indices are m, $\frac{m}{q}$, ..., $\frac{m}{q_1 q_2}$, ..., we should have to replace (m, k), or k, by $k\Pi\left(1-\frac{1}{q}\right)$, just as before we replaced m by $\psi(m)$. But we have to attend to the complete series of columns (P); and thus we have to replace (m, k), not by $k\Pi\left(1-\frac{1}{q}\right)$ taken once, but by $k\Pi\left(1-\frac{1}{q}\right)$ taken as often as there are terms in the product $\Pi\left(1-\frac{1}{r}\right)$, and taken each time with the sign proper to the corresponding term of that product; *i.e.*, (m, k) is replaced by zero.

Lastly, let $k = h\delta$, δ being the greatest common divisor of k and m; so that $(m, k) = (m, \delta) = \delta$. If d is any divisor of m, we have the elementary theorem $\left(\frac{m}{d}, \delta\right) = \left(\frac{m}{d}, h\delta\right)$. For, if $\left(\frac{m}{d}, \delta\right) = \delta'$, we have also $(m, d\delta) = d\delta'$; and hence δ, which is a common divisor of m and $d\delta$, divides $d\delta'$, which is the greatest common divisor of those two numbers. But h is prime to $\frac{m}{\delta}$; therefore, *à fortiori*, h is prime to $\frac{m}{d\delta'}$; *i.e.*, $\left(\frac{m}{d\delta'}, h \cdot \frac{\delta}{\delta'}\right) = 1$, for $\frac{m}{d\delta'}$ is prime to $\frac{\delta}{\delta'}$ as well as to h; or, which is the same thing, $\left(\frac{m}{d}, h\delta\right) = \delta' = \left(\frac{m}{d}, \delta\right)$. It appears from this that in the columns (P) the terms which lie in the row of which the index is $h\delta$, are precisely the same as the terms which lie in the row of which the index is δ; and hence $(m, h\delta)$ is replaced by terms of which the sum is zero, because (m, δ) is replaced by terms of which the sum is zero.

The following remarks are suggested by the preceding theorem, or by its demonstration.

(1.) If we denote by Iα the greatest integer not surpassing the positive quantity α, the theorem may be expressed by the equation

$$\text{(A.)} \quad \frac{\Delta_m}{1 \cdot 2 \cdot 3 \ldots m} = \Pi \left(1 - \frac{1}{p}\right)^{\mathrm{I}\frac{m}{p}},$$

the sign of multiplication extending to all primes not surpassing m.

(2.) Instead of the greatest common divisors themselves, we may consider their powers of exponent s; writing $\Psi_s(m) = m^s \Pi \left(1 - \frac{1}{p^s}\right)$, and following the same course of demonstration, we obtain the theorem

$$\Delta_{m,s} = \Sigma \pm (1, 1)^s (2, 2)^s \ldots\ldots (m, m)^s.$$

$$\text{(B.)} \quad = \Psi_s(1) \times \Psi_s(2) \times \Psi_s(3) \times \ldots\ldots \times \Psi_s(m),$$

or

$$\frac{\Delta_{m,s}}{(1 \cdot 2 \cdot 3 \ldots m)^s} = \Pi \left(1 - \frac{1}{p^s}\right)^{\mathrm{I}\frac{m}{p}};$$

from which we infer, as a particular result,

$$\frac{\Delta_{m,2}}{\Delta_{m,1}} = 1 \cdot 2 \cdot 3 \ldots m \times \Pi \left(1 + \frac{1}{p}\right)^{\mathrm{I}\frac{m}{p}}.$$

(3.) The equation (B) is an identity with respect to the exponent s, which may have any value whatever: the case in which $s = -1$ is especially interesting. Let $[m, n]$ be the least common multiple of m and n, so that $[m, n] = \frac{m \times n}{(m, n)}$; we find

$$\nabla_m = \Sigma \pm [1, 1][2, 2] \ldots\ldots [m, m]$$

$$= 1 \cdot 2 \cdot 3 \ldots m \cdot \Pi (1-p)^{\mathrm{I}\frac{m}{p}},$$

whence

$$\frac{\nabla_m}{\Delta_m} = \pm \Pi \cdot p^{\mathrm{I}\frac{m}{p}};$$

and, in general, if $\nabla_{m,s} = \Sigma \pm [1, 1]^s \ldots\ldots [m, m]^s$,

$$\frac{\nabla_{m,s}}{\Delta_{m,s}} = \pm (\Pi \cdot p^{\mathrm{I}\frac{m}{p}})^s :$$

the sign being that of $(-1)^{\Sigma \mathrm{I}\frac{m}{p}}$.

(4.) If, for the greatest common divisor δ of m and n, we substitute any function whatever $\phi(\delta)$ of δ, and denote by $\Phi(m)$ the function

$$\phi(m) - \Sigma\phi\left(\frac{m}{p}\right) + \Sigma\phi\left(\frac{m}{p_1 p_2}\right) - \ldots\ldots,$$

we arrive at the identity

$$\Sigma \pm \phi(1, 1)\, \phi(2, 2) \ldots\ldots \phi(m, m)$$
$$= \Phi(1) \times \Phi(2) \times \Phi(3) \times \ldots\ldots \times \Phi(m).$$

Two particular cases are worth attention.

(α.) Let s be an integral number, and let

$$\phi_s(m) = 1^s + 2^s + 3^s + \ldots\ldots + (m-1)^s + m^s,$$

so that, when $s > 1$,

$$\phi_s(m) = \frac{m^{s+1}}{s+1} + \tfrac{1}{2}m^s + B_1 \frac{s}{1.2} m^{s-1} - B_3 \frac{s(s-1)(s-2)}{1.2.3.4} m^{s-3} + \ldots\ldots,$$

B_1, B_3, ... being the fractions of Bernoulli, and the last term being $(-1)^{\frac{s-3}{2}} B_{s-2} \frac{s}{2} m^2$, or $(-1)^{\frac{s-2}{2}} B_{s-1} m$, according as s is uneven or even. Let also $\psi_s(m)$ be the sum of the powers s of the numbers prime to m and not surpassing m; we shall have

$$\Sigma \pm \phi_s(1, 1)\ \phi_s(2, 2)\ \ldots\ldots\ \phi_s(m, m)$$
$$= \psi_s(1) \times \psi_s(2) \times \psi_s(3) \times \ldots\ldots \times \psi_s(m, m).$$

The forms of the functions $\psi_s(m)$ are deducible from the expression for $\phi_s(m)$ (see a paper by Mr. Thacker in *Crelle*, Vol. XL., p. 89); we thus find

$$\psi_1(m) = \tfrac{1}{2}m^2 \Pi\left(1 - \frac{1}{p}\right),\ \ \psi_2(m) = \tfrac{1}{3}m^3 \Pi\left(1 - \frac{1}{p}\right) + \tfrac{1}{6}m \Pi(1-a),$$

the general rule being that, in order to obtain $\psi_s(m)$, we are to substitute $m^k \Pi(1 - p^{s-k})$ for m^k in the expression for $\phi_s(m)$.

(β.) Let $\sigma_s(m)$ be the sum of the powers s of the divisors of m; it will be found that

$$\sigma_s(m) - \Sigma\sigma_s\left(\frac{m}{p}\right) + \Sigma\sigma_s\left(\frac{m}{p_1 p_2}\right) - \ldots\ldots = m^s.$$

For, if $m = p_1^{a_1} p_2^{a_2} \ldots\ldots$, and if we put

$$P = (1 + p^s + \ldots\ldots + p^{as}),$$
$$P' = (1 + p^s + \ldots\ldots + p^{(a-1)s}),$$

we have

$$\sigma_s(m) = P_1 \times P_2 \times \ldots\ldots,$$
$$\sigma_s\left(\frac{m}{p_1}\right) = P_1' \times P_2 \times \ldots\ldots,$$
$$\sigma_s\left(\frac{m}{p_1 p_2}\right) = P_1' \times P_2' \times P_3 \ldots\ldots,$$

whence

$$\sigma_s(m) - \Sigma\sigma_s\left(\frac{m}{p}\right) + \Sigma\sigma_s\left(\frac{m}{p_1 p_2}\right) - \ldots\ldots = (P_1 - P_1')(P_2 - P_2') \ldots\ldots = m^s.$$

We thus obtain the equation

$$\Sigma \pm \sigma_s(1, 1)\ \sigma_s(2, 2)\ \sigma_s(3, 3)\ \ldots\ldots\ \sigma_s(m, m) = (1.2.3 \ldots\ldots m)^s,$$

in which s may be any quantity whatever: the cases in which $s=0$, $s=-1$, $s=+1$, are especially remarkable.

(5.) Returning to the equation

$$\begin{aligned}\Delta_m &= \Sigma\pm\,(1,\,1)\,(2,\,2)\,\ldots\ldots\,(m,\,m)\\ &= \psi\,(1)\times\psi\,(2)\times\,\ldots\ldots\,\times\psi\,(m),\end{aligned}$$

we may observe that it is by no means necessary that the numbers 1, 2, 3 ..... m should be the natural series of numbers. We may, in fact, take any different numbers $\mu_1, \mu_2 \ldots\ldots \mu_m$ that we please, subject only to the condition that, if μ be any one of these numbers, every divisor of μ must also appear among them, a condition which implies that unity is always one of the numbers μ. Subject to this condition, we have always

$$\begin{aligned}&\Sigma\pm(\mu_1,\,\mu_1)\,(\mu_2,\,\mu_2)\,\ldots\ldots\,(\mu_m,\,\mu_m)\\ &\quad=\psi\,(\mu_1)\times\psi\,(\mu_2)\,\ldots\ldots\,\psi\,(\mu_m),\end{aligned}$$

or, more generally (see 4 *suprà*),

$$\begin{aligned}&\Sigma\pm\phi\,(\mu_1,\,\mu_1)\,\phi\,(\mu_2,\,\mu_2)\,\ldots\ldots\,\phi\,(\mu_m,\,\mu_m)\\ &\quad=\Phi\,(\mu_1,\,\mu_1)\,\Phi\,(\mu_2,\,\mu_2)\,\ldots\ldots\,\Phi\,(\mu_m,\,\mu_m).\end{aligned}$$

The most obvious cases are—(α) when we reject the multiples of given primes; *e.g.*, when the numbers μ are the uneven numbers in their natural order; (β) when we consider only numbers composed with given primes, *e.g.*, when the numbers are all the divisors of one of them μ; (γ) when we consider only *linear* numbers, *i.e.*, numbers not divisible by any square. In all these cases the results are immediately obtained by the methods which we have already used, and which it is unnecessary to exemplify further.

(6.) Lastly, the symbols μ need not represent integral numbers at all, but may be any quantities which admit of resolution into factors in a definite manner. If, for example, $a^{ij}=x^{i-1}$ or x^{j-1} according as $i<j$, or $j<i$, we have

$$\begin{aligned}&\Sigma\pm a_{11},\,a_{22}\,\ldots\ldots\,a_{mm}\\ &\quad=(x^{m-1}-x^{m-2})\,(x^{m-2}-x^{m-3})\,\ldots\ldots\,(x-1)\\ &\quad=(x-1)^{m-1}x^{\frac{1}{2}(m-1)(m-2)}.\end{aligned}$$

June 8th, 1876.

Prof. H. J. S. SMITH, F.R.S., President, in the Chair.

Dr. Lindemann and Mr. W. S. W. Vaux were present as visitors.

Mr. Kempe spoke on "A general Method of Describing Plane Curves of the nth degree by Linkwork." Prof. Cayley and Mr. Roberts also spoke on the subject. Mr. Roberts then gave an account of a "Further Note on the Motion of a Plane under certain Conditions." Mr. Walker communicated a short Note "On the reduction of the Equation $(U = 0)$ of the Plane Nodal Cubic to its canonical form $ax^3+by^3+6mxyz$." It was shown that the equation to the three lines of reference is $TU+8SH = 0$, and that the equation to (xy) is $L^2-MN = 0$, where L, M, N are the factors of $TU-24SH$. The quotient of $TU+8SH$ by L^2-MN will therefore give the third line z, which is the real axis of inflexion, the other two being the nodal tangents.

Prof. Cayley described a surface, depending upon the sinusoid, which was being constructed for him at Cambridge. The Chairman made a few remarks in connection with M. Hermite's recent Note on a Theorem of Eisenstein's.

On a General Method of describing Plane Curves of the n^{th} degree by Linkwork. *By* A. B. Kempe, B.A.

[*Read June 9th*, 1876.]

Lemma I.—*The Reversor.*—Let $O\xi\beta\alpha$ (Fig. 1) be the linkage known as the contra-parallelogram, $O\xi$ being equal to $\beta\alpha$, and $O\alpha$ to $\xi\beta$.

Make $\alpha\gamma$ a third proportional to $O\xi$ and $O\alpha$, and add the links $O\delta$, equal to $\alpha\gamma$, and $\delta\gamma$, equal to $O\alpha$.

Then the figure $O\alpha\gamma\delta$ is a contra-parallelogram similar to $O\xi\beta\alpha$; and the angle $\xi O\alpha$ is equal to the angle $\delta O\alpha$.

Thus, if $O\xi$ be made to make any angle with $O\alpha$, $O\delta$ will make the same angle with $O\alpha$ on the other side of it.*

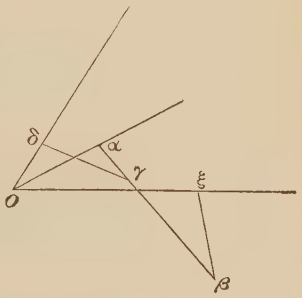

Fig. 1.

* This linkage, and the one next described, were first given by me in the "Messenger of Mathematics," Vol. IV., pp. 122, 123, in a paper "On some new Linkages," §§ 4 and 8.

Lemma II.—*The Multiplicator.*—In the linkage in Lemma I., if $O\alpha$ make an angle θ with $O\xi$, $O\delta$ will make an angle 2θ.

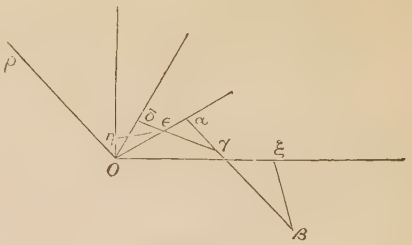

Fig. 2.

Now, if (Fig. 2) we add two links $O\eta$, $\eta\epsilon$ to the linkage $O\alpha\gamma\delta$, in the same manner as we added the links $O\delta$, $\delta\gamma$ to the linkage $O\xi\beta\alpha$, we shall get a link $O\eta$ making an angle 3θ with $O\xi$; and, continuing the construction, we shall get a link $O\rho$ making an angle $r\theta$ with $O\xi$, where r is any integer.

Lemma III.—*The Additor.*—By means of the Reversor (Lemma I.) let the links $O\xi$, $O\pi$ (Fig. 3) be made to make equal angles on each side of the link $O\lambda$. Similarly, let the links $O\sigma$, $O\tau$ be made to make equal angles on each side of $O\lambda$.

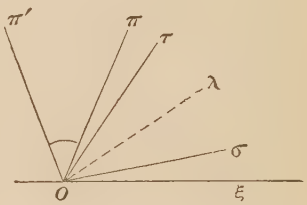

Fig. 3.

Then the angle $\pi O\tau$ equals the angle $\sigma O\xi$; thus, if the angle $\sigma O\xi$ equals $s\theta$ and the angle $\tau O\xi$ equals $r\phi$, the angle $\pi O\xi$ equals $r\phi+s\theta$; or if the angle $\sigma O\xi$ equals $s\theta$ and the angle $\pi O\xi$ equals $r\phi$, then the angle $\tau O\xi$ equals $r\phi-s\theta$. Also, if the link $O\pi'$ be fixed to $O\pi$ (or $O\tau$) so as to make an angle α with it, then the angle $\pi' O\xi$ equals $r\phi+s\theta\pm\alpha$ (or $r\phi-s\theta\pm\alpha$).

Lemma IV.—*The Translator.*—If $O\pi'$ be a link making any angle with $O\xi$ (Fig. 4), and if the two linkage parallelograms $O\pi'\mu\nu$, $\mu\nu\kappa\zeta$ be constructed, it is clear that the link $\kappa\zeta$ is equal and parallel to $O\pi'$, but is otherwise free to assume any position.

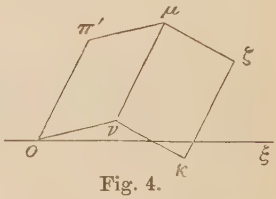

Fig. 4.

Let $\phi(x, y)=0$ be the equation to any plane curve of the n^{th} degree, and let P (Fig. 5) be any point on the curve.

Construct the linkage parallelogram OAPB, in which

$$OA = BP = a,$$
$$OB = AP = b;$$

and let the angle $AOX=\theta$ and the angle $BOX=\phi$.

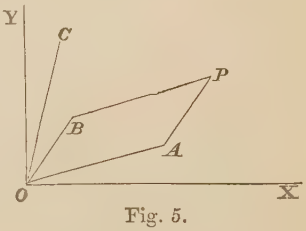

Fig. 5.

Then
$$x = a\cos\theta + b\cos\phi,$$
$$y = a\cos\left(\theta - \frac{\pi}{2}\right) + b\cos\left(\phi - \frac{\pi}{2}\right).$$

Substitute these values of x and y in $\phi(x, y)$, expand and convert powers of cosines into cosines of multiple angles, and then the products of cosines into the cosines of the sums and differences of angles; we shall then get

$$\phi(x, y) = \Sigma\left[\mathrm{A}\cos(r\phi \pm s\theta \pm \alpha)\right] + \mathrm{C},$$

where r and s are positive integers, and $\alpha = \frac{\pi}{2}$ or 0, and A and C are constants.

Now, by means of the Multiplicator and Additor, we may make a link OC make an angle COX, with OX equal to $r\phi \pm s\theta \pm \alpha$.

If then OC equals A, the distance of C from OY equals

$$\mathrm{A}\cos(r\phi \pm s\theta \pm \alpha).$$

Make then a series of links OC_1, OC_2, &c. make angles with OX equal to the angles of the expression $\Sigma[\ ]$, and let their lengths be the coefficients of the corresponding cosines. Then the sum of the distances of C_1, C_2, &c. from OY will equal

$$\Sigma\left[\mathrm{A}\cos(r\phi \pm s\theta \pm \alpha)\right].$$

If then, by means of the Translator, we construct the chain of links OP_1, P_1P_2, &c. (Fig. 6), each link of which is parallel and equal to one of the links OC_1, OC_2, &c.; the point P′, the extremity of the chain, will be at a distance from OY equal to

$$\Sigma\left[\mathrm{A}\cos(r\phi \pm s\theta \pm \alpha)\right] = \phi(x, y) - \mathrm{C}.$$

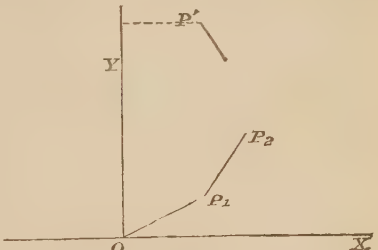

Fig. 6.

But since P lies on the curve, therefore $\phi(x, y) = 0$;

therefore P′ lies at a constant distance $-\mathrm{C}$ from OY; *i.e.*, it describes the straight line
$$x + \mathrm{C} = 0.$$

Conversely, if P′ be made by any of the known "exact parallel motions" to describe the straight line
$$x + \mathrm{C} = 0,$$
P will describe the curve $\phi(x, y) = 0$.

It is hardly necessary to add, that this method would not be practically useful on account of the complexity of the linkwork employed, a necessary consequence of the perfect generality of the demonstration.

The method has, however, an interest, as showing that there *is* a way of drawing any given case; and the variety of methods of expressing particular functions that have already been discovered renders it in the highest degree probable that in every case a simpler method can be found. There is still, therefore, a wide field open to the mathematical artist to discover the simplest linkworks that will describe particular curves.

The extension of this demonstration to curves of double curvature and surfaces clearly involves no difficulty.

Further Note on the Motion of a Plane under certain Conditions.

By Samuel Roberts.

[*Read June 9th*, 1876.]

In my paper "On the Motion of a Plane under certain Conditions" (Proc., Vol. III., p. 286 *et seq.*), the movement was determined by the paths of two of the points of the plane over a coincident stationary plane. I propose now to consider briefly two other elementary cases to which I referred at the close of my former paper, and relative to which I had then obtained some general results.

Those cases are: (1) when the motion is determined by the path of a point of the moving plane and by the envelope of one of its lines; (2) when the motion is determined by the envelopes of two of the lines of the plane.

(A.) *Line-directrix and Point-directrix.*

(1.) *Locus of Points.*

In the first case the problem is to determine the characteristics of the locus of a point P carried by a constant angle ABC moving in its plane, so that while the side AB always touches a fixed curve, which I call the line-directrix, the other side moves with a fixed point C always on another fixed curve, which I call the point-directrix. The two directrices are supposed to lie in the same plane as the moving angle.

I consider in the first instance a simple case, namely, when the line-directrix is a point, and the point-directrix is a straight line.

The locus is a circular curve of the fourth order and sixth class, having usually two finite double points and another at infinity, where the point-directrix meets the line at infinity. The line-directrix is the focus formed by the circular asymptotes.

If, however, the apex B moves on the point-directrix, then the locus

of a point on AB (produced as may be necessary) is a conchoid of Nicomedes, and one finite node passes to infinity and joins there the other infinitely distant node. The same takes place for all points on a line through C parallel to AB. (Proc., Vol. II., p. 128.)

Suppose now that we take n points for our line-directrix and m' lines for our point-directrix, the order of the compound locus is $4nm'$ and its class is $6nm'$.

The corrections necessary to obtain the general order and class have to be investigated.

If a multiple point of the order p exists on the point-directrix at C, it follows from the nature of the description that the corresponding position of P, the describing point, determines on the locus a multiple point of the order p. There are $2n$ such positions of P corresponding to $2n$ tangents common to the line-directrix and a circle about C as centre. That is to say, the multiple point C determines $2n$ multiple points on the locus. Again, if there is a tangent to the line-directrix touching it at p points, there are $2m'$ positions of P corresponding to this multiple tangent, and each position determines a multiple point of the order p.

It may be inferred also from the nature of the description that these multiple points have the same number of multiple tangents that the corresponding multiple points on the point-directrix have of coincident tangents, or the corresponding multiple tangents of the line-directrix have of coincident contacts, as the case may be. In short, the singularities produced are similar individually to those producing them. Unless there are special relations existing in the construction, a case which is now excluded, these are the only singularities produced by singularities of the directrices. With regard then to our special directrices, we have to correct for $\frac{n(n-1)}{2}$ double tangents passing through the points of the line-directrix taken two and two, and for $\frac{m'(m'-1)}{2}$ intersections of the m' lines of the point-directrix; but, if further we suppose that there remain τ double tangents, ι inflexions on the line-directrix, and δ' nodes, κ' cusps on the point-directrix, we must correct by deduction.

Thus we obtain for the general directrices:

$$\text{Order of locus} \qquad 4nm',$$

$$\text{class of locus} \qquad 6nm' + 2n(n-1)m' + 2m'(m'-1)n - 4\tau m' - 6\iota m' - 4\delta' n - 6\kappa' n,$$

$$\text{or} \qquad 6nm' + 2mm' + 2nn'.$$

The only cusps are those introduced by the inflexions and cusps of the line and point-directrices respectively in number

$$2\iota m' + 2\kappa' n.$$

The circular points are multiple in the order nm'; the m' points, in which the point-directrix meets the line at infinity, are multiple in the order $2n$.

It must be always remembered that special positions of P relative to the traversing angle may occasion an alteration of the singularities. So also special relations of the angle to the directrices, or of the directrices to each other, are excluded from consideration.

Even in the general case, the multiple points not due to singularities on the directrices are of a complex character. For instance, it has been remarked that, for a point as line-directrix and a line as point-directrix, the circular asymptotes intersect in the line-directrix, in other words, in the focus of the line-directrix. But the point singularities at infinity are not in general affected by finite singularities of the directrices. Hence the multiple circular points have coincident tangents in consequence of contiguous multiple points. There are n real foci of the line-directrix. Through each of these will pass m' lines meeting the locus at a circular point in $nm'+m'$ consecutive points. In fact the nm'-fold circular point is accompanied by n contiguous points multiple in the order m'.

In like manner, the loci of points on a line through C parallel to BA will have, at the points where the point-directrix meets the line at infinity, m' pairs of contiguous multiple points of the order $2n$.

It is useful to regard the motion in different points of view. For instance, if we draw a line through P parallel to BA and join PC, the locus is evidently the same as that of the apex of a constant angle, one side of which touches a parallel of the original directrix, while the other moves with a fixed point on the point-directrix. Again, if we draw a line through C parallel to BA, it appears that the same motion is produced by making the apex of the angle move along the point-directrix, while one side envelopes a parallel of the original line-directrix.

The locus of a point in BA (continued if necessary) is the same as that of the apex of a constant angle moving in the same manner relative to the directrices.

The question suggests itself, whether the reduction of the gross order, which must take place when we substitute a parallel of the line-directrix for the line-directrix itself, arises from the singularities of the parallel or otherwise. It appears that this effect is produced by the breaking up of the locus into line components, one of which is the locus required, the other corresponding to a different line BA.

To shew this, let the line-directrix be such that its parallel breaks up as indicated by the tangential equation

$$(\alpha^2+\beta^2)\,\varphi^2+\psi^2=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (a),$$

where φ, ψ are functions of the order $n-1$ and n respectively in α, β.

It will suffice to suppose that the moving angle is a right angle, so that we have, for three of the conditions, say,

$$ax+\beta y-1=0,\quad \beta(x-x')-\alpha(y-y')=0,\quad (x-x')^2+(y-x')^2=l^2,$$

giving $\alpha^2+\beta^2=\dfrac{(ax'+\beta y'-1)^2}{l^2}$, and (a) breaks up into two conditions. Hence, if the line-directrix is such that its parallel breaks up, the locus of the apex of an angle or other point rigidly connected with it also breaks up.

But the parallel of the parallel of a curve necessarily breaks up. Therefore the gross order $4\,.\,2n\,.\,m'$ must be divided by 2. Hence also we see that the multiple circular points on the line-directrix do not reduce the order, when at least the point-directrix is not circular.

If, in like manner, we take the class with reference to the parallel of the original line-directrix, we have

$$6\,.\,2nm'+2\,(2n+2m)\,m'+2\,.\,2nn';$$

and the class of one component is

$$6nm'+2mm'+2nn'+2nm',$$

to be diminished, as I shall presently shew, by $2nm'$, on account of multiple circular points.

Circular Directrices.

If we take a point as the line-directrix and a circle as the point-directrix, the locus of a point carried by the angle is of the sixth order, the circular points are triple points, each having a pair of coincident tangents which intersect the conjugate pair at the centre of the fixed circle. There are three other nodes. The remaining tangents at the triple points intersect in the fixed point, and the two singular foci and the nodes lie in a circle. This is a case of the three-bar motion curve in which two of the three singular foci coincide, and the actual description by means of three links is impracticable.

The equation may be obtained as follows:—

Let ABC be the angle whose tangent is a constant t, and which moves so that BA passes through the origin of coordinates while C moves on a circle whose equation is $(x-a)^2+y^2=r^2$.

Let P be the describing point, and draw PD perpendicular to BC. Then, if PD$=l$, CD$=p$, BD$=q$, and BC makes the angle ϕ with the axis of x, and if x_1, y_1 are the coordinates of B, x_2, y_2 those of C, and X, Y those of P, we have

$$y_1=\frac{\sin\phi-t\cos\phi}{\cos\phi+t\sin\phi}x_1,\qquad (x_2-a)^2+y_2^2=r^2,$$

$$x_1=\mathrm{X}-l\sin\phi-q\cos\phi,\quad x_2=\mathrm{X}-l\sin\phi-p\cos\phi,$$
$$y_1=\mathrm{Y}+l\cos\phi-q\sin\phi,\quad y_2=\mathrm{Y}+l\cos\phi-p\sin\phi,$$

from which we get

$$(X-Yt)\sin\phi-(Y+Xt)\cos\phi+qt-l=0,$$
$$\{pY+l(X-a)\}\sin\phi+\{p(X-a)-lY\}\cos\phi + \frac{r^2-l^2-p^2-(x-a)^2-y^2}{2}=0;$$

and finally, writing Φ for $\frac{r^2-l^2-p^2-(x-a)^2-y^2}{2}$,

$$\{(Y+Xt)\Phi+(qt-l)[p(X-a)-lY]\}^2 + \{(X-Yt)\Phi-(qt-l)[pY+l(X-a)]\}^2 - \{(p+lt)(X^2+Y^2)-al(Xt+Y)-ap(X-Yt)\}^2=0.$$

Writing this in the form

$$[L\Phi+(qt-l)M]^2+\{P\Phi-(qt-l)Q\}^2-\{LQ+MP\}^2=0,$$

we see that the circular points are triple, and the three usually finite nodes lie on a circle through the origin and the centre of the fixed circle.

To find the tangents at the circular points, we may substitute α for $\frac{X+iY}{2}$, β for $\frac{X-iY}{2}$, then the coefficient of β^3 is the same as in

$$(1+t^2)(X^2+Y^2)\Phi^2+[k(X^2+Y^2)+l_1]\Phi+(mX^2+Y^2+n_1)=0.$$

where k and m are constant, and l_1, n_1 are of the first order in X, Y. We have for the tangents

$$\alpha(2a-a)^2=0, \quad \text{or} \quad (X+iY)[(X-a)+iY]^2=0,$$

shewing that there are two pairs of coincident tangents intersecting at the centre of the fixed circle, and that the remaining pairs intersect at the origin.

Again, if we take a circle as line-directrix and another circle as point-directrix, we have a locus which breaks up into two curves of the same nature as the one just discussed.

If we take as line-directrix a circle and as point-directrix a straight line, the locus consists of two curves of the fourth order and sixth class, of the same kind as when we take a point and line as directrices. Although the loci break up, we are obliged to take the components into account, because the locus does not break up in the first of these elementary cases.

When, therefore, the line-directrix consists of $n-2p$ points and p circles, and the point-directrix consists of $m'-y$ lines and q circles, the order of the general locus is

$$4(n-2p)(m'-2q)+6(n-2p)q+8(m'-2q)p+12pq,$$

or

$$4nm'-2nq,$$

and no correction is necessary on account of the breaking up of the directrices.

Again, for the class we have

$$6(n-2p)(n'-2q)+10(n-2p)q+12(m'-2q)p+20pq,$$

and we must add the correction for nodes and double tangents (leaving however the two multiple circular points of the point-directrix),

$$4m'.\frac{n-2p\,.\,n-2p-1}{2}+4n\frac{m'-2q\,.\,m'-2q-1}{2}$$
$$+4m'.2p(n-2p)+4n.2q(m'-2q)\quad-4nq(q-1)$$
$$+4m'.2p(p-1)+4n.2q(q-1).$$

And if there are τ double tangents and ι inflexions on the line-directrix and δ' double points, κ' cusps on the point-directrix, we have over corrected, and must now deduct

$$4m'\tau+6m'\iota+4[\delta'-q(q-1)]n+6n\kappa',$$

so that finally, for the general class, we have

$$6nm'+2mm'+2n'n-6nq-4m'p.$$

The circular points are multiple in the order $nm'+nq$, and the $m'-2q$ points at infinity not circular of the point-directrix are of the order $2n$.

There are at the circular points q groups of $2n$ coincident tangents intersecting their conjugates at the q^2 singular foci of the point-directrix, and $(n-2p)$ groups of $(m'-q)$ coincident tangents intersecting at the $(n-2p)^2$ ordinary foci of the line-directrix, and p groups of $2(m'-q)$ coincident tangents intersecting their conjugates at the p^2 singular foci of the line-directrix.

These coincident tangents are not altogether cuspidal, but are in great part due to adjacent multiple parts.

There remains the equivalent of $2nq$ cusps. We have further the $2\iota m'+2\kappa' n$ cusps introduced by singularities of the directrices.

The line-directrix touches the line at infinity.

When the line-directrix touches the line at infinity, there is a reduction of order and class.

For, take as line-directrix $n-p$ finite points and p at infinity, and as point-directrix m' lines. Then the order of the general locus is

$$4(n-p)m'+2pm'=(4n-2p)m',$$

and no correction is necessary.

For the class we have, correcting as before,

$$6(n-p)m'+2[m(m-1)-p(p-1)]m'+2m'(m'-1)m$$
$$-4\left(\tau-\frac{p(p-1)}{2}\right)m'-6\iota m'-4\delta' n-6\kappa' n.$$

giving $$6(n-p)m'+2mm'+2n'n.$$

The modifications in the multiple points at infinity may be traced as

before. The circular points are of the order $(n-p)\,m'$ and the m' points at infinity of the point-directrix are still of the order $2n$. There are at a circular point $n-p$ groups of m' coincident tangents intersecting their conjugates in the $(n-p)^2$ foci of the line-directrix. For simplicity's sake, I have treated the several cases successively. We might have combined them with no great difficulty in accordance with the hypothesis that the directrices are both circular and parabolic.

(2.) *Envelopes of Lines.*

There is in general a one-to-one correspondence between the points of the locus and the lines of the envelope. When the line-directrix is a point and the point-directrix is a line, the envelope of a line, taken generally, is of the fourth class and sixth order, having four double points and six cusps, and being therefore unicursal. It is also circular. Hence, if we take n points for the line-directrix and m' lines for the point-directrix, the class is $4n'm$, which holds good generally. When a multiple point is produced in the locus of a point on the traversing line by a singularity of the directrices, a corresponding multiple tangent is given to the envelope. Hence, for the general order we get, by correcting as previously,

$$6nm' + 2mm' + 2nn'.$$

And we may trace the nature of the singularities at infinity to some extent. Thus there is a multiple tangent of the order $2nm'$, or, at least, we have its equivalent while the circular parts are multiple in the order nm'.

The circumstances of the general case are, however, more readily comprehended when we consider the envelope of a line passing through C, while B coincides with C, so that the apex moves on the point-directrix. When we take a point and line for directrices, as before, the envelope of the line through C is a parabola. This gives us for the corresponding class in the more general case $2nm'$, and for the order $2nm' + nm' + nm'$. And the envelope of a line taken generally is the parallel of such an envelope for a line through C; that is to say, it is of the class $4nm'$, and the order

$$4nm' + 2(2nm' + mm' + n'n) - 2nm' = 6nm' + 2mm' + 2nn',$$

the number $2nm'$ being deducted from the gross order in consequence of the tangencies at infinity. The characteristics of the general envelope are therefore known by the theory of parallels. The circular points are of the order nm', and there are nm' contiguous pairs of nodes at infinity, accounting in this case for the $2nm'$-fold tangent. The other $nm' + n'm$ points at infinity are double.

If, in the general case, we take the envelope of a line through C, it is still of the class $4nm'$, &c.; but it is of such a form that its parallel

breaks up, and the two components correspond to different lines of the plane.

If the line-directrix touches the line at infinity, we have reductions corresponding to those of loci in the like case; namely, the class is $(4n-2p)\,m'$, and the order $6(n-p)\,m'+2mm'+2\iota'n$. In the special case when B and C coincide and the line passes through C, the class is $(2n-p)\,m'$, and the order $2(n-p)\,m'+mm'+n'n$.

I suppose that we might obtain the values of the order and class of envelopes in the case of circular directrices by the method used in the case of loci.

(B.) *Two Line-Directrices.*

(1.) *Locus of Points.*

The second case mentioned at the beginning of this paper requires us to determine the characteristics of the locus of a point carried in its plane by a constant angle whose sides touch two directrices.

It will be shorter to take at once the circular directrices, for which purpose we have to consider the case of two points as directrices. The locus of the apex is here a circle; that of any other point is a limaçon. Next we will take a point and a circle as directrices. The locus of the apex is a limaçon; that of a point not the apex consists of two limaçons. Again, if the directrices are two circles, the locus of the apex consists of two limaçons; that of a point not the apex consists of four limaçons.

Consider, then, two directrices consisting respectively of $n-2p$ points, p circles; and $n'-2q$ points, and q circles.

The order of the locus described by the apex is

$$2(n-2p)(n'-2q)+4(n-2p)\,q+4(n'-2q)\,p+8pq=2nn',$$

which requires no correction in the general case.

For the class we have

$$2(n-2p)(n'-2q)+4(n-2p)\,q+4(n'-2q)\,p+8pq=2nn';$$

and the correction is, if we continue the notation previously used,

$$\begin{aligned}&\frac{2n-2p\,.\,n-2p-1}{2}+\frac{2n'-2q\,.\,n'-2q-1}{2}\\&+4p(p-1)\,n'+4q(q-1)\,n\\&+4(n-2p)\,pn'+4(n'-2q)\,qn\\&-2\tau n'-3\iota n'-2\tau' n-3\iota' n;\end{aligned}$$

and the general class is $2nn'+n'm+nm'$.

The circular points are multiple in the order nn', and there are cuspidal coincident tangents as well as tangents otherwise coincident.

If the describing point is not at the apex, the class and order are doubled. The circular points are also cuspidal in a higher degree.

If the directrices touch the line at infinity, there are further reductions. Thus, if the directrices touch that line r and s times respectively, we have for the order of the apex locus

$$2(n-r)(n'-s)+r(n'-s)+s(n-r)=2nn'-rn'-sn;$$

and the class is

$$2(n-r)(n'-s)+(n'-s)m+(n-r)m'.$$

We may evidently consider the locus of a general point as described by the apex of a constant angle moving about two parallels of the directrices. But, in order to make use of this apparently more simple conception, we have to take account of the singularities of the parallels, particularly at infinity. If two directrices are such that their parallels break up, the locus of the apex breaks up.

(2.) *Envelopes of Lines.*

For two points as directrices the envelope of a line through the apex is a point on the circle described by the apex.

By taking a system of n points for one line-directrix, and n' points for the other, we get for the class generally nn'.

The order is the correction

$$\frac{2n'(n'-1)}{2}n'+\frac{2n(n-1)}{2}n'$$
$$-2\tau n'-3\iota n'-2\tau' n-3\iota' n,$$

or

$$nm'+n'm.$$

For a line not passing through the apex we must have the parallel of a corresponding envelope by a line through the apex.

The class is $2nn'$, and the order

$$2nn'+2nm'+2n'm.$$

The circular points are now multiple in the order nn', and the $nm'+n'm$ points, in which the curve enveloped by a parallel line through the apex meets the line at infinity, are double.

If the directrices touch the line at infinity r and s times respectively, the class of the apex envelope becomes

$$(n-r)(n'-s)+r(n'-s)+s(n-r)=nn'-rs,$$

and the order is

$$(n-r)m'+(n'-s)m.$$

I infer from these formulæ, what I dare say is otherwise known, that the oblique pedal of a curve is the ordinary pedal of a curve of the same order and class.

I have not discussed some important modifications and special rela-

tions, as, for instance, the coincidence of directrices. Special relations between the directrices as common foci, &c. augment in a notable manner the difficulty of determining the characteristics. But such relations exist in full force when there is only one directrix, and I am not able to give satisfactory results.

In the first case, the order of the general locus is not changed when the directrices coincide; that is, it is $4nm$. In the second case, it is $4n(n-1)$ for a general point, $2n(n-1)$ for the apex; but if the angle is a right angle, the latter locus is of the order $n(n-1)$, and I presume the first order is also reduced. In these instances it is the class which offers considerable difficulty.

Notes on the Communication entitled "On the Transformation of Elliptic Functions." By Prof. W. K. Clifford, M.A., F.R.S.

Some of the following notes * would have been incorporated in the paper by the process of revision for the press, if that had not been kindly performed for me during an enforced absence from the English climate. As regards all but one of them, I am glad of the opportunity which has thus been afforded me of extension and correction. But it is a matter of great regret to me that I discovered too late the priority of M. Darboux in the principal theorem of the second part of the paper; viz., the porismatic character of a polygram circumscribed to a conic and totally inscribed in a curve of order one less than the number of sides. In one of the notes to a book which it is almost inexcusable in a geometer not to have read, marked, learned, and inwardly digested,† M. Darboux has stated and proved the theorem, and has followed it by further investigations of the highest interest and importance. The method even of my investigation is the same as that of M. Darboux (as indeed was inevitable from the nature of the subject), namely, the representation of a point in a plane by means of the parameters of the tangents drawn from it to a fixed conic. It is not the first time that I have had the honour of following, however distantly, in the footsteps of that eminent geometer; but on other occasions it has been my good fortune to discover the fact in time.

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* These remarks apply also to certain developments which I have since thought it better to communicate under separate titles.

† "Sur une classe remarquable de courbes et de surfaces algébriques." Paris, Gauthier-Villars, 1873. Note II., p. 183.

Completion of the Geometric Proof of the Transformation-Formulæ.

In my former paper one point was assumed as given by the analytical theory of transformation, namely, that the new argument u' is equal to the old argument u divided by a constant M. Having now found a simple proof of this, I will take the liberty of re-stating in outline, for the sake of clearness, the whole demonstration; availing myself of a remark of M. Darboux.

It is proved by Jacobi's method that if $x_1 x_2 \ldots x_n$ are parameters of the points of contact of the n sides of a polygon circumscribed to a conic U and inscribed in a conic V, and if ± 1, $\pm k^{-1}$ are the parameters of the points of contact of the common tangents of the two conics, then

$$x_1 = \operatorname{sn} u,\ x_2 = \operatorname{sn}(u+\gamma),\ x_3 = \operatorname{sn}(u+2\gamma),\ \ldots\ x_n = \operatorname{sn}[u+(n-1)\gamma],$$

where $n\gamma = 4\mathrm{K}+4i\mathrm{K}'$, and the modulus of the elliptic function is k.

This being so, an infinite number of in-and-circumscribed polygons can be drawn.

If $p_n = 0$ be the equation in x whose roots are $x_1 x_2 \ldots x_n$, and $q_n = 0$ an equation in x whose roots are the parameters of the sides of another such polygon; then $p_n - y q_n = 0$ will have for its roots the parameters of the sides of an in-and-circumscribed polygon, whatever value be given to y.

For the locus of the $\frac{1}{2}n\overline{n-1}$ intersections of the tangents at the n points $p_n - y q_n = 0$, when y is made to vary, is a curve of order $n-1$, which has $2n$ points in common with the conic V, and therefore contains that conic entirely.*

The quantity y being now regarded as the parameter of a varying polygon, let p_n be chosen to represent that polygon which has the parameter of one of its sides equal to zero, and q_n to represent that which has the parameter of one of its sides infinite. Then y, which is $p_n : q_n$, will be an odd function of x, because of the symmetry of the figure and the fact that $q_n = 0$ (having one root infinite) is only of degree $n-1$.

If then p_n and q_n be affected with such constant multipliers that $y = 1$ when $x = 1$, we must have $y = -1$ when $x = -1$. And we may suppose that $y = \pm\lambda^{-1}$ when $x = \pm k^{-1}$.

Now the intersections of corresponding sides of two in-and-circumscribed polygons lie on a conic touching the common tangents of U and V.

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* This is in substance the remark of M. Darboux referred to. Op. cit., p. 190.

For the parameters being respectively $\operatorname{sn} u$, $\operatorname{sn}(u+\gamma)$, &c., and $\operatorname{sn} v$, $\operatorname{sn}(v+\gamma)$, &c., the common difference of the arguments is $u-v$.

If we suppose the two polygons to vary subject to this condition, their parameters y and η will be connected by a 2, 2 correspondence, such that the values of y which make the two corresponding values of η coincide are ± 1, $\pm\lambda^{-1}$.

Therefore, if $y = \operatorname{sn}(u', \lambda)$, we must have $\eta = \operatorname{sn}(u'+c', \lambda)$, where c' is a constant. But the relation between corresponding sides x, ξ, of these polygons is $x = \operatorname{sn}(u, k)$, $\xi = \operatorname{sn}(u+c, k)$, since they intersect on the fixed conic V.

Hence the quantities u, u' are so related that a constant difference between two values of u implies a constant difference between the corresponding values of u'. Hence * (by Euclid's definition of proportion) a varying difference between two values of u implies a *proportional* difference between the corresponding values of u'. But $u' = 0$ when $u = 0$; therefore the two quantities are proportional, and $u = \mathrm{M}u'$ where M is constant.

Now the relation between the parameters of two consecutive sides of a polygon being of the second degree in each, the products x_2x_n, x_3x_{n-1}, &c., are given as ratios of quadratic functions of x_1. Hence the product $x_1x_2 \ldots x_n$, regarded as a function of x_1, is a rational fraction whose numerator is of order n, and whose denominator is of order $n-1$. But y is a rational fraction whose numerator and denominator are of just these orders; and y vanishes whenever one of the quantities $x_1x_2 \ldots x_n$ vanishes, and becomes infinite when one of them becomes infinite.

Therefore $$y = mx_1x_2x_3 \ldots\ldots x_n,$$

where m is a constant; that is to say,

$$\operatorname{sn}\left(\frac{u}{\mathrm{M}}, \lambda\right) = m \operatorname{sn} u \operatorname{sn}(u+\gamma) \operatorname{sn}(u+2\gamma) \ldots\ldots \operatorname{sn}[u+(n-1)\gamma],$$

when $n\gamma = 4\mathrm{K}+4i\mathrm{K}'$. We determine m by remarking that $y = 1$, when $x = 1$, or when $u = \mathrm{K}$; and then λ, by remarking that $y = \lambda^{-1}$, when $x = k^{-1}$, or when $u = \mathrm{K}+i\mathrm{K}'$. Finally M is determined by differentiating the equation and making $u = 0$.

(Cayley's Theorem.)—Every Cubic Transformation has an Elliptic Differential which it transforms.

This theorem was given by Prof. Cayley in the Philosophical Magazine, Vol. 15 (Fourth Series), p. 363. I here reproduce his investigation, slightly altered to suit the generalization which follows. On the very

* This is Archimedes' proof that a body which passes over equal spaces in equal times will pass over proportional spaces in unequal times.

beautiful solution of the complete question (Given the elliptic differential, to find the transformation) by Hermite (Crelle, vol. 60, p. 304) and Clebsch (Theorie der binären alg. Formen, p. 405), I hope to say something at another time.*

Let U, V be any two cubic functions of x, and consider the transformation $y = \frac{U}{V}$.

Suppose that

$$\text{Disct.}\,(U - Vy) = A + By + Cy^2 + Dy^3 + Ey^4,$$

where, of course, A = Disct. U, E = Disct. V. And let $y_1 y_2 y_3 y_4$ be the roots of the equation Disct. $(U - Vy) = 0$. Then each of the cubics $U - Vy_1$, $U - Vy_2$, $U - Vy_3$, $U - Vy_4$ has a square factor, because its discriminant vanishes. Now, if $U - Vy_1$ has a square factor $(x-\alpha)^2$, then $x - \alpha$ divides $U' - V'y_1$; that is, for the value $x = \alpha$ we have at the same time

$$\begin{matrix} U - Vy_1 = 0 \\ U' - V'y_1 = 0 \end{matrix} \qquad \left[U', V' = \frac{dU}{dx}, \frac{dV}{dx}\right],$$

and therefore also $VU' - V'U = 0$; that is to say, those four linear factors which are squared in the cubics $U - Vy_1$, &c., occur as single factors in $VU' - V'U$. It follows that

$$A\,(U - Vy_1)(U - Vy_2)(U - Vy_3)(U - Vy_4) = (VU' - V'U)^2 \,.\, X,$$

where X represents the product of the single factors of the four cubics. Or, which is the same thing,

$$AU^4 + BU^3V + CU^2V^2 + DUV^3 + EV^4 = (VU' - V'U)^2\, X.$$

Now, since $y = \frac{U}{V}$, we have $dy = \frac{VU' - V'U}{V^2}\, dx$. Hence, if we transform the differential $\frac{dy}{\sqrt{\text{Disct.}\,(U - Vy)}}$ by the substitution $y = \frac{U}{V}$, we get

$$\frac{VU' - V'U}{V^2} \cdot \frac{V^2}{(VU' - V'U)\sqrt{X}}\, dx \text{ or } \frac{dx}{\sqrt{X}}.$$

That is, we have $$\frac{dy}{\sqrt{\text{Disct.}\,(U - Vy)}} = \frac{dx}{\sqrt{X}},$$

where y is connected with x by the equation $y = \frac{U}{V}$, which is the theorem in question.

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* The foot-note in my previous paper gave an erroneous expression for X. The article there referred to (Gundelfinger, Math. Annalen., Vol. VII., p. 452) is a simplification of the method and results of Clebsch in regard to the typical representation of two cubics.

New Stand-Point for the Algebraic Transformation-Theory.

We may generalize this result by applying an analogous treatment to transformations of any order. The problem is considered in the following form: Given a transformation $y = U : V$, it is required to find—

(1) What are the necessary and sufficient conditions to be satisfied by the functions U, V, in order that the transformation $y = U : V$ may be able to transform an elliptic differential;

(2) These conditions being supposed satisfied, what is the differential which can thus be transformed.

We will consider first the case in which U and V are of odd order, say $2m+1$, or, to speak more correctly, the case in which $U - Vy$ is of order $2m+1$ in x.

The necessary conditions may at once be derived from consideration of the varying in-and-circumscribed polygon the parameters of whose sides are the roots of the equation $U - Vy = 0$.

Starting with any one side of the polygon, which touches what we may call the inner conic, we find its intersections with the outer conic, and then from these draw new tangents to the inner conic. Proceeding in this way symmetrically on both sides of the original tangent, we find at last that the two tangents to the inner conic meet on the same point of the outer conic. We must clearly end with two tangents, and not with one, because the polygon has an odd number of sides.

We might, however, start with a vertex of the polygon on the outer conic, draw two tangents to the inner conic, then from their intersection with the outer conic two more tangents, and so on: at last we shall reach a pair of vertices such that the line joining them touches the inner conic. In this case we must end with two vertices on one side, not with one vertex on two sides, for the same reason as before, that we have supposed the polygon to have an odd number of sides.

Now suppose that in the first mode of construction we start with a common tangent to the two conics; then its two intersections with the outer conic will coincide, and consequently the tangents from them to the inner conic coincide also. We may, however, go on with the construction; and, after drawing $m+1$ successive tangents, we shall have an exceptional case of an in-and-circumscribed polygon, in which the side first drawn (the common tangent of the two conics) counts singly, and each of the m others counts doubly, so that the polygon has altogether $2m+1$ sides. But the last pair of tangents being coincident, must be regarded as intersecting on the inner conic; and therefore their point of contact must be an intersection of the two conics.

So that we cannot by the second mode of construction get a degenerate polygon of an odd number of sides different from those just considered. If we start with a vertex at a point of intersection of the two conics, the tangents drawn from this to the inner conic will of course coincide, and so therefore will the points in which they meet the outer conic again, and we may continue the process; but we only have a right to stop when the line joining the two last coincident vertices on the outer conic (*i. e.*, the tangent at that point to the outer conic) touches the inner conic; that is to say, when we come upon a common tangent of the two conics.

What actually happens may be illustrated by the case of an in-and-circumscribed pentagon. Let ab be a common tangent of the two conics, touching the inner conic at a, and the outer at b. From b draw the other tangent bc to the inner conic, meeting the outer conic again at c; then from c draw the other tangent cd to the inner conic meeting the outer conic again at d. Then, *if pentagons can be drawn inscribed to the outer conic, and circumscribed to the inner, the point d will be an intersection of the two conics.* And the pentagon whose sides are dc, cb, bab, bc, cd is a degenerate case of an in-and-circumscribed pentagon; the side bab being single, and the sides bc, cd, each of them double.

Observe that, if d were not an intersection of the two conics, we should still have an improper solution of the problem, to find five points on the outer conic such that the line joining every successive two shall touch the inner conic. But if we started from d as an intersection of the conics, and then found the points c, b, a as before, except that ba is now not a tangent to the outer conic at b, we should not have found a solution of that problem, but of this other—To find five tangents to the inner conic, so that the intersection of every successive two shall be upon the outer conic.

For the purpose of our present investigation, the result may be stated thus: an in-and-circumscribed polygon of an odd number of sides can only have two sides coincident, when one of its sides is a common tangent of the two conics, and all the others coincide in pairs.

Or, the equation $\mathrm{U}-\mathrm{V}y=0$ can only have coincident roots, when one represents a common tangent, and the others coincide in pairs: so that $\mathrm{U}-\mathrm{V}y$ has one single factor, and m square factors. Moreover, there are four values of y, and four only, which bring this about. Now the discriminant of $\mathrm{U}-\mathrm{V}y$ is of the order $4m$. Hence *we must have*

$$\text{Disct. } (\mathrm{U}-\mathrm{V}y) = \mathrm{Y}^m, \text{ where } \mathrm{Y} = (1, y)^4,$$

and if $y_1 y_2 y_3 y_4$ *are the roots of the equation* $(1; y)^4 = 0$, *then each of the quantities* $\mathrm{U}-\mathrm{V}y_1$, *&c., has one single factor and m square factors.*

These conditions are necessary; we shall shew that they are sufficient, by solving the second part of the problem.

In the first place, we have

$$(1, 0)^4 . (U - Vy_1)(U - Vy_2)(U - Vy_3)(U - Vy_4) = (U, V)^4,$$

and therefore, as before,

$$(U, V)^4 = (VU' - V'U)^2 X,$$

where $(VU' - V'U)^2$ is the product of all the square factors, and X of the four single factors. From this it follows immediately that

$$\frac{dy}{\sqrt{Y}} = \frac{dx}{\sqrt{X}},$$

if $y = \frac{U}{V}$; which is the transformation required. The result may thus be stated :—

The necessary and sufficient conditions that the substitution $y = U : V$ shall be able to transform an elliptic differential, $U - Vy$ being of order $2m+1$ in x, are that Disct. $(U - Vy)$ shall be a perfect mth power, and that those forms $U - Vy$ which have a square factor at all shall have m square factors. This being so, the differential $\frac{dy}{\sqrt[2m]{\text{Disct. } (U - Vy)}}$ will be transformed by the given substitution into $\frac{dx}{\sqrt{X}}$, where X is the product of the four single factors.

It may be observed that, if those forms $U - Vy$ which have a square factor at all have m square factors, it will follow that Disct. $(U - Vy)$ is a perfect mth power; but the converse is not true.

Passing now to the case of a transformation of even order, we enquire, as before, in what cases the in-and-circumscribed polygon can have two sides coincident. If we start with a tangent to the inner conic, and from its intersections with the outer conic draw two more tangents, and so on; there cannot be an in-and-circumscribed polygon of an even number of sides, unless we come to a pair of intersections such that the line joining them touches the inner conic. Suppose then that the first side is a common tangent, so that its two intersections with the outer conic coincide, and that we draw another tangent from this point, another from its second point of intersection, and so on; we must finally come to a point on the outer conic where the tangent touches the inner conic; that is, we must come to another common tangent. In the case of a quadrilateral, for example, let ab be a common tangent, touching the inner conic at a and the outer at b. From b draw the other tangent to the inner conic, meeting the outer again at c; then cd must be also a common tangent, touching the outer conic at c and the inner at d. Thus the sides of the degenerate quadrilateral are bab, bc, cdc, cb, the sides bab, cdc counting singly, and bc double. And in general the two common tangents will count singly, and all the

rest double. It is manifest that there are only two degenerate polygons of this kind, each containing two of the four common tangents.

The second construction also gives us two degenerate polygons, but of quite a different character. Starting with a point on the outer conic, we draw two tangents to the inner, and from their new intersections with the outer, two more, and so on; we must at last come to two tangents which meet on the outer conic. If then our starting-point is a point of intersection, so that the two tangents coincide all through the process, we must come to a pair whose intersection, that is, their point of contact with the inner conic, is on the outer conic; or, which is the same thing, we must come to another intersection of the conics. To use again the quadrilateral as an illustration, the tangents to the inner conic at two points of intersection α and γ must meet on the outer conic at β, and the sides of the quadrilateral are then $\alpha\beta$, $\beta\gamma$, $\gamma\beta$, $\beta\alpha$, so that *all* of them count double. And generally, in degenerate polygons of this kind, all the sides count double. There are clearly two such degenerate polygons, each having two points of intersection for vertices.

To sum up, then, there are four degenerate polygons of even order $2m$; two of them have each two common tangents as sides, and two of them have each two points of intersection as vertices. The former have the common tangents as single sides, and all the rest double; the latter have all their sides double.

It follows that, if the substitution $y = \mathrm{U} : \mathrm{V}$ is capable of transforming an elliptic differential, $\mathrm{U}-\mathrm{V}y$ being of order $2m$ in x, there are only four values of y which make $\mathrm{U}-\mathrm{V}y$ have a square factor; two of these make it have m square factors, and the other two make it have $m-1$ square factors and two single factors. Consequently the former two are m-fold roots, and the latter two $(m-1)$-fold roots, of the equation $\mathrm{Disct.}\,(\mathrm{U}-\mathrm{V}y) = 0$. That is to say, we have

$$\mathrm{Disct.}\,(\mathrm{U}-\mathrm{V}y) = \mathrm{Y}^{m-1}.\,(y-y_1)(y-y_2),$$

where $$\mathrm{Y} = k\,(y-y_1)(y-y_2)(y-y_3)(y-y_4) = (1, y)^4, \text{ say.}$$

Hence, as before, $$(\mathrm{U}, \mathrm{V})^4 = (\mathrm{VU}'-\mathrm{V}'\mathrm{U})^2.\,\mathrm{X},$$

where X is the product of the four single factors due to y_3 and y_4. From these equations it follows directly that

$$\frac{dy}{\sqrt{\mathrm{Y}}} = \frac{dx}{\sqrt{\mathrm{X}}}$$

which is the transformation required.

In regard to these conditions it is to be observed that in general they imply a special constitution of the quantics U, V, as well as a special relation of them to each other. This consideration, however, does not

come in until the sixth order of transformation is reached. Thus, in the case of the quartic transformation, the only condition is that U, V shall be simultaneously reducible to the canonical form; which being so, we may find linear combinations of them such that one is the Hessian of the other, thus falling back upon Hermite's very elegant form. In the quintic transformation U may be taken arbitrarily, but the involution $U - Vy$ is then completely determined. In the sextic transformation, however, U and V must each be the product of three quadrics in involution (viz., the same involution in the two cases); so that a certain invariant of each must vanish. (Salmon's "Higher Algebra," p. 210 and Appendix; see Clebsch, "Alg. Formen," p. 298.)

The following presents were made to the Society in the Recess:—

"Monatsbericht," Ap., Mai, Juni, 1876.

"Proceedings of the Royal Society," Vol. xxv., Numbers 168, 170, 171, 172, 173.

"Crelle," 81er Band, viertes Heft; 82er Band, 1es, 2es Heft.

"Jahrbuch über die Fortschritte der Mathematik," sechster Band, Jahrgang, 1874, Heften 1, 2, 3; Berlin, 1876.

"Proceedings of the Physical Society," Vol. ii., Part 1," Nov. 1875 to June 1876.

"Bulletin des Sciences Mathématiques," Tome x^{me}, Av., Mai, Juin, 1876.

"Reprint from 'Educational Times,'" Vol. xxv.

"Truth in Extremis—a Plea against the Murder of Science by the Gold Poison," by a Searcher; Oxford, 1876.

"Transactions of the Connecticut Academy of Arts and Sciences," Vol. iii., Part 1; New Haven, 1876.

"Vierteljahrsschrift der naturforschenden Gesellschaft in Zürich," neunzehnter Jahrgang, Heften 1—4, 1874; zwanzigster Jahrgang, Heften 1—4, 1875.

"Sulle Soluzioni singolari delle Equazioni alle derivate parziali," Nota del Prof. Felice Casorati.

"Ueber lineare Differentialgleichungen," Prof. Klein, 26 Juni, 1876, (from the Erlangen *Transactions*).

"Société des Sciences physiques et naturelles de Bordeaux—Extrait des Procès-Verbaux des Séances," Bordeaux.

"Washington Astronomical and Meteorological Observations, 1873, at the United States Naval Observatory," Rear-Admiral Sands; Washington, 1875.

"Educational Times," July—Nov. 1876.

"Reale Istituto Lombardo di Scienze e Lettere—Memorie," Vol. xiii.—iv. della Serie iii., Fasc. ii., 1875.

"Reale Istituto Lombardo di Scienze e Lettere — Rendiconti," Serie ii.; Vol. vii., Fasc. xvii—xx, 12 Nov. 1874 — 24 Dic. 1874, Milano, 1874; Vol. viii., Fasc. 1—20, 7 Gennajo—23 Dicembre 1875, Milano, 1875.

"Mathematische Annalen," Extract, pp. 446—546, Vol. x.: H. G. Zeuthen "Sur la théorie des surfaces réciproques," Copenhague, 27 Avril, 1876.

"Mathematische Annalen," Extract, p. 365—397: F. Klein "Ueber Abelsche Integrale," München, April, 1876.

"Mathematische Annalen," Extract, p. 398—416: F. Klein "Ueber Riemannsche Flächer," München, April, 1876.

"Proceedings of the Musical Association for the investigation and discussion of subjects connected with the Art and Science of Music," Second Session," 1875–6.

"Bulletin de la Société Mathématique de France," Tome iv., Mai, No. 5, Paris, 1876.

"Elementary Treatise on Elliptic Functions," by Prof. Cayley, F.R.S.

"Proceedings of the Royal Irish Academy," Vol. ii., Series ii., Nos. 4 (Oct. 1875), 5 (Jan. 1876), 6 (July, 1876).

"Transactions of the Royal Irish Academy, Vol. xxvi.: Science, 1, Jan. 1876; 2, 3, March, 1876; 4, May, 1876; 5, June, 1876; Vol. xxiv., Part ii., Polite Literature.

Royal Irish Academy, List of Members, &c., 31st July, 1876.

"Catalogue of the Books in the Library of the Manchester Literary and Philosophical Society," Manchester, 1875.

"Proceedings of the Literary and Philosophical Society of Manchester," Vol. xv., Session 1875-76.

"Memoirs of the Literary and Philosophical Society of Manchester," Third Series, 5th Volume.

W. Bretschneider, "Ueber Curven vierter Ordnung mit drei Doppelpunkten," Stuttgart, 1875.

L. Wedekind, "Beiträge zur geometrischen Interpretation binärer Formen," Erlangen, 1875.

W. Braun, "Die Singularitäten der Lissajous'chen Stimmgabelcurven," Erlangen, 1875.

Dr. Paul Gordan, "Ueber das Formen-System binärer Formen," Leipzig, 1875.

Axel Harnack, "Ueber die Verwerthung der elliptischen Functionen für die Geometrie der Curven dritten Grades," Leipzig, 1875.

APPENDIX.

PROF. SYLVESTER has not presented to the Society any account of the results got in his communication "On the Fifteen Young Ladies Problem" (p. 9). The question was originally proposed by the Rev. T. P. Kirkman, in the "Lady's and Gentleman's Diary" for 1850: in the following *Diary* two solutions were given. A more complete and systematic solution was given by Mr. Woolhouse (the editor) in the *Diary* for 1862, under the title "On Triadic Combinations of 15 Symbols." A paper on the same problem was published in the "Quarterly Journal of Pure and Applied Mathematics," No. 31, 1867, in which Mr. J. Power, M.A., Fellow of Clare College, Cambridge, discussed some results arrived at by Messrs. Cape and Horner, and gave his own solution. Prof. Sylvester, in the "Educational Times" for November, 1875, proposed the question in the following shape: "In a school of 15 girls, a rule has been laid down that they shall walk out every day in rows of threes, but that the same two girls shall never come together twice in the same row. The rule is supposed to have been carried out correctly during the six working days of the week, but when the time comes for their going to church together on Sunday, it is found to be absolutely impossible to continue it any further. Investigate whether the rule *can* have been correctly carried out during the six previous days; and, if so, shew how." No answer to the question has been given in the "Educational Times." With reference to this subject, Mr. S. M. Drach, F.R.A.S., writes, under date Nov. 18, 1875: "As the School Girl problem is again mooted before the Mathematical Society (proposed by Mr. Kirkman, *Diary*, 1850, answered 1851—and expatiated on by Mr. Woolhouse, "Assurance Magazine," Oct. 1862), I find there are 455 sole triads, viz.,

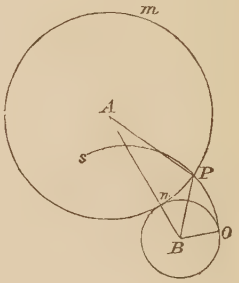
$$\text{ABC, ABD, ABE, ... } = 91, \quad \text{BCD, BCE, ... } = 78.$$

In all,

$$91 + 78 + 66 + 55 + 45 + 36 + 28 + 21 + 15 + 10 + 6 + 3 + 1 = 455$$
$$= 5 \times 91 = \tfrac{5}{4} \text{ year.}$$

"Perhaps the following illustration which I invented on 15th January, 1863, may make a nice Christmas Gift if artistically executed." *

There was a slight error in Major Campbell's paper, which occurred almost at the very commencement of it. This vitiated his subsequent reasoning. The following short notice of it will be sufficient to give an idea of the subject of the communication.



OPs is the curve of a cam turning on an axis B. It intersects in P a circle mPn, described by the end of a lever AP on which the cam acts, and when P is at n the commencement O of the curve is at n. The ratio of the angular velocity of AP round A to that of BP round B is a given function, $\mathrm{F}(\theta)$, of the angle ABO $(=\theta)$ described by the cam. Required the equation to OPs; B being the pole, and BO (considered at rest) the base of the curve.

Let $\quad \mathrm{AP} = a$, $\mathrm{AB} = b$, $\mathrm{BP} = r$, and $\angle \mathrm{OBP} = \theta'$.

Then
$$\angle \mathrm{BAP} = \int \mathrm{F}(\theta)\, d\theta,$$

and
$$r = \pm\sqrt{a^2+b^2-2ab\cos\int \mathrm{F}(\theta)\, d\theta}.$$

From the last equation we obtain

$$\theta = f_1(r),$$

$$\angle \mathrm{ABP} = \cos^{-1}\frac{r^2+b^2-a^2}{2br} = f_2(r),$$

Hence $\theta' = f_1(r) - f_2(r)$ represents the general equation. In few

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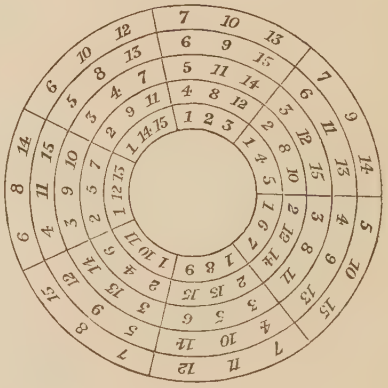
* The diagram was somewhat of the accompanying form, but finished off so as to represent a rose (title, Filiarum Rosa Scholastica Effigies). Directions were given for giving to the several days of the week different colours, and in the central space were placed a church, houses, trees, birds, &c.: *church* emblemizes theology; *trees*, botany; *cows*, *birds*, natural history; *gravel walks*, geology; and so on. Names were also assigned to the fifteen young ladies, and they were awarded prizes alliteratively; as, Bashful Bessie, prize for Botany.



cases can it be reduced to any simple form; but, by giving different values to r, we may obtain from it (by the help of a table of cosines) the corresponding values of θ', and thus trace the curve.

From the fact of its possible application to mechanism, the case when $F(\theta) =$ a constant might merit investigation.

The following abstracts of Prof. Smith's three communications (p. 28) have appeared in the "Messenger of Mathematics," No. lvii., January 1876 (pp. 143, 144):

On a Problem of Eisenstein's.—If p is an uneven prime, the function $4\frac{x^p-1}{x-1} = Z$ can always be expressed in the form $Y^2-(-)^{\frac{1}{2}(p-1)}pX^2$, where X and Y are rational and integral functions of x, having integral coefficients. This is a theorem of Gauss.

Eisenstein's problem (Crelle's Journal, vol. xxvii., p. 83) is, "To determine the cases in which the equation $Z = Y^2-(-)^{\frac{1}{2}(p-1)}pX^2$ admits of a multiplicity of solutions, and to ascertain the law connecting the various solutions, when there is more than one." The solution of this problem is as follows: if [T, U] is *any* solution whatever in integral numbers of the equation $T^2-(-)^{\frac{1}{2}(p-1)}pU^2 = 4$, and [X, Y] is any one given solution of Gauss's equation, then all the solutions of Gauss's equation are comprised in the formula

$$[\tfrac{1}{2}\{TX+(-)^{\frac{1}{2}(p-1)}pUV\},\ \tfrac{1}{2}(UX+TY)].$$

Thus, if $p = 4n+3$, the equation admits of but one solution (the four solutions $[\pm X, \pm Y]$ being regarded as but one), except in the case $p = 3$, when it admits of three; if $p = 4n+1$, the equation admits of an infinite number of solutions. That the functions

$$[\tfrac{1}{2}(TX+pUY),\ \tfrac{1}{2}(UX+TY)]$$

are all of them solutions of Gauss's equation, is evident: the proof that this formula comprises all the solutions of the equation is less elementary, because it depends on the irreducibility of the function Z. There exists a general theory of the representation of rational and integral functions of x by quadratic forms; such representation being, of course, only possible when the given function of x is capable of resolution into two factors by the adjunction of a quadratic surd.

On the Joint Invariants of two Conics or two Quadrics.—Let P and Q be two conics, and let 123 be any triangle self-conjugate with regard to P. Let also P_1, P_2, P_3 be the rectangles of the points 1, 2, 3 with regard to the conic P, these rectangles being taken upon transversals measured in any fixed direction, and let Q_1, Q_2, Q_3 have similar meanings with regard to the conic Q, the direction of the transversals being

also fixed. Then the expression $\frac{Q_1}{P_1}+\frac{Q_2}{P_2}+\frac{Q_3}{P_3}$ has the same value for all self-conjugate triangles of P, and is, in fact, that invariant of P, Q which is linear with regard to Q and quadratic with regard to P, and the evanescence of which expresses that Q harmonically circumscribes P. The corresponding theorem in the geometry of the straight line is, "If Q_1Q_2, P_1P_2 are two pairs of fixed points on a line, and if A_1A_2 is any pair of harmonic conjugates of P_1P_2, the value of the expression $\frac{A_1Q_1 \cdot A_1Q_2}{A_1P_1 \cdot A_1P_2}+\frac{A_2Q_1 \cdot A_2Q_2}{A_2P_1 \cdot A_2P_2}$ is independent of the particular pair, A_1A_2, considered." From this theorem the result given above for two conics follows immediately: from it the corresponding property for two quadrics may be inferred, viz., $\frac{Q_1}{P_1}+\frac{Q_2}{P_2}+\frac{Q_3}{P_3}+\frac{Q_4}{P_4}=\text{constant}$; and so on for quadratic functions containing any number of indeterminates.

On the Equation $P \times D = constant$, *of the Geodesic Lines of an Ellipsoid.*—From this equation (in which P is the perpendicular from the centre upon the tangent plane at any point of the geodesic, and D is the semi-diameter parallel to the tangent line of the geodesic), it is convenient to be able to infer directly the principal properties of the geodesic line, without having first to transform the equation into M. Liouville's form $\mu^2 \cos^2 \iota + \nu^2 \sin^2 \iota = a^2$. In Dr. Salmon's *Geometry of Three Dimensions*, the theorem of the constancy of the sum or difference of the geodesic radii vectores, drawn from any point of a line of curvature to two umbilics, is thus demonstrated. And it is worth while to add (though it is very improbable that the point has not been noticed before) that a proof of the theorem, that two geodesic tangents of a line of curvature, which intersect at right angles, intersect on a sphero-conic, may similarly be obtained without transforming the equation. Let Q be the point where the two geodesic tangents intersect at right angles, O the centre of the ellipsoid; let $c = OQ$, and let a, b be the semi-axes of the central section parallel to the tangent plane at Q. The two geodesics make angles of 45° with the lines of curvature at Q; hence, for either of these geodesic lines, $D^2 = \frac{2a^2b^2}{a^2+b^2}$. Let Q′ be a second point where two geodesic tangents to the same line of curvature intersect at right angles: then $\frac{2P^2a^2b^2}{a^2+b^2} = \frac{2P'^2a'^2b'^2}{a'^2+b'^2}$, because $P \times D$ has the same value for all geodesic lines touching the same line of curvature. But $P^2a^2b^2 = P'^2a'^2b'^2$, because parallelepipeds circumscribing an ellipsoid with their faces parallel to conjugate diametral planes are equal. Hence $a^2+b^2 = a'^2+b'^2$. But also

$$a^2+b^2+c^2 = a'^2+b'^2+c'^2,$$

therefore $c = c'$, and Q and Q′ lie on the same sphero-conic.

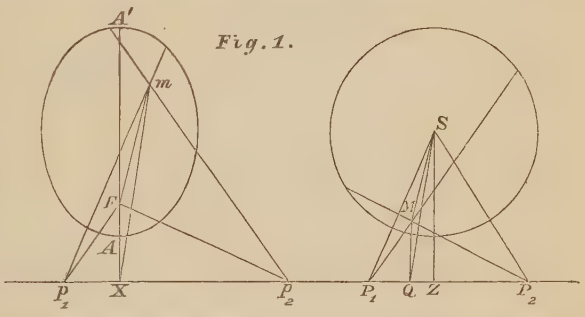
The title of Prof. Sturm's paper referred to on p. 194 is "Das Problem der Collineation," Math. Ann., Vol. x, Heft 1, pp. 117—137.

The method of Mr. Renshaw's proof (p. 195) will be seen from the following account of it:—

The inscription of a Polygon in a Circle, subject to the condition that "each of its sides shall pass through a given point," having already been accomplished by several eminent geometers, and especially by the late Mr. Swale of Liverpool, in a remarkably easy manner;—the object of the present paper is to show how by an easy transformation, effected by means of the generating circle, the construction of the problem in the circle can be rendered available to the resolution of the same problem in the conic sections. Before proceeding to show how this is done, it will be necessary to premise the following:—

LEMMA.—If m be any point in the plane of a conic, focus F and directrix XZ, X being the foot of the axis,—then, if Fm, Xm be joined, and from S, the centre of a generating circle, parallels be drawn to Fm, Xm, the latter of which (SQ) meets the directrix in Q, and QM be drawn perpendicular to the directrix and meeting the first parallel in M, the point M in the circle will be reciprocal to the point m in the conic.

Fig. 1.



For, draw through F any two lines meeting the directrix in p_1, p_2, and join mp_1, mp_2; also draw MP_1, MP_2 parallel respectively to Fp_1 Fp_2, and meeting the directrix in P_1, P_2, and join SP_1, SP_2. Then, by similar triangles, mF : FX : Fp_1 = SM : MQ : MP_1, and mF, Fp_1 are respectively parallel to SM, MP_1, therefore mp_1 is parallel to SP_1, similarly mp_2 is parallel to SP_2.

Hence the lines mp_1, mp_2 are reciprocal respectively to MP_1, MP_2; and therefore the points m and M are reciprocal in the conic and the circle. Wherefore, conversely, if the points m and M be reciprocal, the triangles FmX, MSQ are similar, and have their homologous sides respectively parallel.

COR.—Hence, if a point m be given in the plane of a conic, it is evident how to obtain its reciprocal M in a generating circle.

In order to apply the preceding to the solution of the proposed prob-

lem—Let k, l, m, n, o, be given points in the plane of a conic, focus F, directrix XZ, and S the centre of a generating circle.

Join Fk, Fl, Fm, Fn, Fo, Xk, Xl, Xm, Xn, Xo and draw through S parallels to them, the latter five lines meeting the directrix in Q_1, Q_2, Q_3, Q_4, Q_5, and through these points draw perpendiculars to the directrix meeting the parallels to Fk, Fl, ... Fo, through S, in K, L, M, N, O. Now inscribe in the circle a polygon whose sides pass through K, L, M, N, O, and let its sides produced meet the directrix in P_1, P_2, P_3, P_4, P_5, and join these points with S; through F draw parallels to the sides of the polygon inscribed in the circle, and let them meet the directrix in p_1, p_2, p_3, p_4, p_5; lastly, draw through p_1, p_2 ..... p_5 parallels to FP_1, FP_2 ..... FP_5; these lines shall form a polygon inscribed in the conic, and passing through the given points k, l, m, n, o. This will be evident from the principle of the generating circle and from what has been premised.*

In the solution of the problem for the circle, Mr. Swale has made it to depend upon the Theorem, "That if any number of chords AB, BC, CD, DE, ... FG be drawn in a circle, parallel to as many fixed lines, the line (AG) joining the first and last points, will be given in position." Hence, taking Mr. Swale's figure (page 45, Liverpool Apollonius, q. v.), and constructing the reciprocal figure for the Conic Sections, we find a like Theorem holds also for those curves. For if *abcdehg* be the figure in a conic, and reciprocal to ABCDEHG in a generating circle pertaining thereto; then, if the sides produced of the figures in the conic and circle meet the directrix in the points $p_1, p_2 \ldots p_7$, P_1, P_2 ... P_7; then, since in the circle the last line AG is fixed (by Swale's Theorem), and AG is parallel to Fp_7, therefore Fp_7 is fixed; consequently in the conics the Theorem becomes the following,—that, "If in any conic any number of chords *ab*, *bc*, *cd*, *de* ... *hg* be drawn, and produced to meet the directrix in p_1, $p_2 \ldots p_6$, so that Fp_1, Fp_2 ... Fp_6, when joined, are parallel to fixed lines; then, if *ag* be joined, and produced to meet the directrix in p_7, Fp_7, being joined, will also be parallel to a fixed line.

Mr. Swale has also shown (p. 45 and 46, *ibid.*) that the angular points of a polygon inscribed in a circle, and whose sides pass through fixed points, lie on fixed lines. Hence we may conclude, reciprocally, that the same holds good in the case of any conic. For, if we construct the polygon in the conic by the means proposed in this paper, then, since the angles of the reciprocal polygon inscribed in the gene-

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* The points k, l, m, n, o may have any given positions whatever, all lying in the same right line or not. Figures were drawn by Mr. Renshaw to illustrate the case of a Pentagon inscribed in an Ellipse, and that of a Quadrilateral inscribed in an Hyperbola.

rating circle lie on fixed lines (Swale), on the reciprocals of these fixed lines, which will also themselves be fixed, will lie the angles of the conic—that is, the angles of the conic lie on fixed lines.

Mr. J. J. Walker's communication is printed in the "Quarterly Journal of Pure and Applied Mathematics," No. 55, Oct. 1876 (pp. 242—245), under the title "On the Equation of the Nodal Plane Cubic." In the same number of the "Quarterly Journal" (pp. 229—235) is given an account of the Sextic Torse referred to by Prof. Cayley (p. 213 *supra*). Mr. Hanlon's paper on "The Contracted Vein" (Proceedings, Vol. iii., pp. 4, 5) is referred to in a paper by Lord Rayleigh. ("Notes on Hydrodynamics." Phil. Mag., Dec. 1876.)

At an early period in the season an application was made to the Council by the Lords of the Committee of Council on Education, in reply to which application it was resolved that Plücker's models of Curved Surfaces should be exhibited at the Loan Exhibition of scientific apparatus at the South Kensington Museum. The models were carefully renovated and in many cases corrected by M. Nolet, of University College, under the superintendence of Prof. Henrici, F.R.S. It was Prof. Henrici's intention to have given an account of the construction of these and of some of his own models at one of the morning meetings at the Museum; he was, however, obliged to abandon the idea. An account of the models was printed in the catalogue, the basis of the description being the paper by Prof. Cayley on the models (Proceedings, Vol. iii., pp. 281—285), supplemented by a description of models A, B, C, D, E, F, drawn up by Prof. Henrici. The models attracted some attention, and were referred to in the "Handbook," * and also in several of the accounts drawn up for the daily and other papers and journals.

R. T.

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* Handbook to the Special Loan Collection of Scientific Apparatus, 1876. "Geometrical Instruments and Models," by Prof. H. J. S. Smith, F.R.S., pp. 34—54.

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